

3.46 For the vector field $\mathbf{E} = \hat{\mathbf{x}}xz - \hat{\mathbf{y}}yz^2 - \hat{\mathbf{z}}xy$, verify the divergence theorem by computing:

- (a) the total outward flux flowing through the surface of a cube centered at the origin and with sides equal to 2 units each and parallel to the Cartesian axes, and
- (b) the integral of $\nabla \cdot \mathbf{E}$ over the cube's volume.

Solution:

- (a) For a cube, the closed surface integral has 6 sides:

$$\begin{aligned}\oint \vec{E} \cdot d\vec{s} &= F_{\text{top}} + F_{\text{bottom}} + F_{\text{right}} + F_{\text{left}} + F_{\text{front}} + F_{\text{back}}, \\ F_{\text{top}} &= \int_{x=-1}^1 \int_{y=-1}^1 (\hat{\mathbf{x}}xz - \hat{\mathbf{y}}yz^2 - \hat{\mathbf{z}}xy) \Big|_{z=1} \cdot (\hat{\mathbf{z}} dy dx) \\ &= - \int_{x=-1}^1 \int_{y=-1}^1 xy dy dx = \left(\left(\frac{x^2 y^2}{4} \right) \Big|_{y=-1}^1 \right) \Big|_{x=-1}^1 = 0, \\ F_{\text{bottom}} &= \int_{x=-1}^1 \int_{y=-1}^1 (\hat{\mathbf{x}}xz - \hat{\mathbf{y}}yz^2 - \hat{\mathbf{z}}xy) \Big|_{z=-1} \cdot (-\hat{\mathbf{z}} dy dx) \\ &= \int_{x=-1}^1 \int_{y=-1}^1 xy dy dx = \left(\left(\frac{x^2 y^2}{4} \right) \Big|_{y=-1}^1 \right) \Big|_{x=-1}^1 = 0, \\ F_{\text{right}} &= \int_{x=-1}^1 \int_{z=-1}^1 (\hat{\mathbf{x}}xz - \hat{\mathbf{y}}yz^2 - \hat{\mathbf{z}}xy) \Big|_{y=1} \cdot (\hat{\mathbf{y}} dz dx) \\ &= - \int_{x=-1}^1 \int_{z=-1}^1 z^2 dz dx = - \left(\left(\frac{xz^3}{3} \right) \Big|_{z=-1}^1 \right) \Big|_{x=-1}^1 = \frac{-4}{3}, \\ F_{\text{left}} &= \int_{x=-1}^1 \int_{z=-1}^1 (\hat{\mathbf{x}}xz - \hat{\mathbf{y}}yz^2 - \hat{\mathbf{z}}xy) \Big|_{y=-1} \cdot (-\hat{\mathbf{y}} dz dx) \\ &= - \int_{x=-1}^1 \int_{z=-1}^1 z^2 dz dx = - \left(\left(\frac{xz^3}{3} \right) \Big|_{z=-1}^1 \right) \Big|_{x=-1}^1 = \frac{-4}{3}, \\ F_{\text{front}} &= \int_{y=-1}^1 \int_{z=-1}^1 (\hat{\mathbf{x}}xz - \hat{\mathbf{y}}yz^2 - \hat{\mathbf{z}}xy) \Big|_{x=1} \cdot (\hat{\mathbf{x}} dz dy) \\ &= \int_{y=-1}^1 \int_{z=-1}^1 z dz dy = \left(\left(\frac{yz^2}{2} \right) \Big|_{z=-1}^1 \right) \Big|_{y=-1}^1 = 0, \\ F_{\text{back}} &= \int_{y=-1}^1 \int_{z=-1}^1 (\hat{\mathbf{x}}xz - \hat{\mathbf{y}}yz^2 - \hat{\mathbf{z}}xy) \Big|_{x=-1} \cdot (-\hat{\mathbf{x}} dz dy)\end{aligned}$$

$$= \int_{y=-1}^1 \int_{z=-1}^1 z \, dz \, dy = \left(\left(\frac{yz^2}{2} \right) \Big|_{z=-1}^1 \right) \Big|_{y=-1}^1 = 0,$$

$$\oint \vec{E} \cdot d\vec{s} = 0 + 0 + \frac{-4}{3} + \frac{-4}{3} + 0 + 0 = \frac{-8}{3}.$$

(b)

$$\begin{aligned} \iiint \nabla \cdot \vec{E} \, dv &= \int_{x=-1}^1 \int_{y=-1}^1 \int_{z=-1}^1 \nabla \cdot (\hat{\mathbf{x}}xz - \hat{\mathbf{y}}yz^2 - \hat{\mathbf{z}}xy) \, dz \, dy \, dx \\ &= \int_{x=-1}^1 \int_{y=-1}^1 \int_{z=-1}^1 (z - z^2) \, dz \, dy \, dx \\ &= \left(\left(\left(xy \left(\frac{z^2}{2} - \frac{z^3}{3} \right) \right) \Big|_{z=-1}^1 \right) \Big|_{y=-1}^1 \right) \Big|_{x=-1}^1 = \frac{-8}{3}. \end{aligned}$$
