3.48 A vector field $\mathbf{D} = \hat{\mathbf{r}}r^3$ exists in the region between two concentric cylindrical surfaces defined by $r = 1$ and $r = 2$, with both cylinders extending between $z = 0$ and $z = 5$. Verify the divergence theorem by evaluating:

(a) $\oint_S \mathbf{D} \cdot d\mathbf{s}$,

(b) $\int_{\mathscr{V}} \nabla \cdot \mathbf{D} \, d\mathscr{V}$.

Solution:

(a)

$$\iint \vec{D} \cdot d\vec{s} = F_{\text{inner}} + F_{\text{outer}} + F_{\text{bottom}} + F_{\text{top}},$$

$$F_{\text{inner}} = \int_{\phi=0}^{2\pi} \int_{z=0}^{5} \left((\hat{\mathbf{r}}r^3) \cdot (-\hat{\mathbf{r}}r \, dz \, d\phi) \right) \Big|_{r=1}$$

$$= \int_{\phi=0}^{2\pi} \int_{z=0}^{5} \left(-r^4 \, dz \, d\phi \right) \Big|_{r=1} = -10\pi,$$

$$F_{\text{outer}} = \int_{\phi=0}^{2\pi} \int_{z=0}^{5} \left((\hat{\mathbf{r}}r^3) \cdot (\hat{\mathbf{r}}r \, dz \, d\phi) \right) \Big|_{r=2}$$

$$= \int_{\phi=0}^{2\pi} \int_{z=0}^{5} \left(r^4 \, dz \, d\phi \right) \Big|_{r=2} = 160\pi,$$

$$F_{\text{bottom}} = \int_{r=1}^{2} \int_{\phi=0}^{2\pi} \left((\hat{\mathbf{r}}r^3) \cdot (-\hat{\mathbf{z}}r \, d\phi \, dr) \right) \Big|_{z=0} = 0,$$

$$F_{\text{top}} = \int_{r=1}^{2} \int_{\phi=0}^{2\pi} \left((\hat{\mathbf{r}}r^3) \cdot (\hat{\mathbf{z}}r \, d\phi \, dr) \right) \Big|_{z=5} = 0.$$

Therefore, $\iint \vec{D} \cdot d\vec{s} = 150\pi$.

(b) From the back cover, $\nabla \cdot \vec{D} = (1/r)(\partial/\partial r)(rr^3) = 4r^2$. Therefore,

$$\iiint \nabla \cdot \vec{D} \, d\mathscr{V} = \int_{z=0}^{5} \int_{\phi=0}^{2\pi} \int_{r=1}^{2} 4r^2 r \, dr \, d\phi \, dz = \left(\left(\left(r^4 \right) \Big|_{r=1}^{2} \right) \Big|_{\phi=0}^{2\pi} \right) \Big|_{z=0}^{5} = 150\pi.$$