3.51 Repeat Problem 3.50 for the contour shown in Fig. P3.50(b).

Solution: In addition to the independent condition that $z = 0$, the three lines of the triangle are represented by the equations $y = 0$, $y = 2 - x$, and $y = x$, respectively.

(a)

$$\oint \vec{E} \cdot d\vec{l} = L_1 + L_2 + L_3,$$

$$L_1 = \int \left(\hat{\mathbf{x}}xy - \hat{\mathbf{y}}\left(x^2 + 2y^2\right)\right) \cdot \left(\hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz\right)$$

$$= \int_{x=0}^{2} (xy)|_{y=0,z=0}\,dx - \int_{y=0}^{0} \left(x^2 + 2y^2\right)\big|_{z=0}\,dy + \int_{z=0}^{0} (0)|_{y=0}\,dz = 0,$$

$$L_2 = \int \left(\hat{\mathbf{x}}xy - \hat{\mathbf{y}}\left(x^2 + 2y^2\right)\right) \cdot \left(\hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz\right)$$

$$= \int_{x=2}^{1} (xy)|_{z=0,y=2-x}\,dx - \int_{y=0}^{1} \left(x^2 + 2y^2\right)\big|_{x=2-y,z=0}\,dy + \int_{z=0}^{0} (0)|_{y=2-x}\,dz$$

$$= \left(x^2 - \frac{x^3}{3}\right)\bigg|_{x=2}^{1} - \left(4y - 2y^2 + y^3\right)\big|_{y=0}^{1} + 0 = \frac{-11}{3},$$

$$L_3 = \int \left(\hat{\mathbf{x}}xy - \hat{\mathbf{y}}\left(x^2 + 2y^2\right)\right) \cdot \left(\hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz\right)$$

$$= \int_{x=1}^{0} (xy)|_{y=x,z=0}\,dx - \int_{y=1}^{0} \left(x^2 + 2y^2\right)\big|_{x=y,z=0}\,dy + \int_{z=0}^{0} (0)|_{y=x}\,dz$$

$$= \left(\frac{x^3}{3}\right)\bigg|_{x=1}^{0} - \left(y^3\right)\big|_{y=1}^{0} + 0 = \frac{2}{3}.$$

Therefore,

$$\oint \vec{E} \cdot d\vec{l} = 0 - \frac{11}{3} + \frac{2}{3} = -3.$$

(b) From Eq. (3.105), $\nabla \times \vec{E} = -\hat{\mathbf{z}}3x$, so that

$$\iint \nabla \times \vec{E} \cdot d\vec{s} = \int_{x=0}^{1} \int_{y=0}^{x} \left((-\hat{\mathbf{z}}3x) \cdot (\hat{\mathbf{z}}\,dy\,dx)\right)\big|_{z=0}$$

$$+ \int_{x=1}^{2} \int_{y=0}^{2-x} \left((-\hat{\mathbf{z}}3x) \cdot (\hat{\mathbf{z}}\,dy\,dx)\right)\big|_{z=0}$$

$$= -\int_{x=0}^{1} \int_{y=0}^{x} 3x\,dy\,dx - \int_{x=1}^{2} \int_{y=0}^{2-x} 3x\,dy\,dx$$

$$= -\int_{x=0}^{1} 3x\,(x - 0)\,dx - \int_{x=1}^{2} 3x\,((2 - x) - 0)\,dx$$

$$= -\left(x^3\right)\Big|_0^1 - \left(3x^2 - x^3\right)\Big|_{x=1}^2 = -3.$$