3.55 Verify Stokes's theorem for the vector field $\mathbf{B} = (\hat{\mathbf{r}} \cos\phi + \hat{\boldsymbol{\phi}} \sin\phi)$ by evaluating:

(a) $\oint_C \mathbf{B} \cdot d\boldsymbol{\ell}$ over the path comprising a quarter section of a circle, as shown in Fig. P3.55, and

(b) $\int_S (\nabla \times \mathbf{B}) \cdot d\mathbf{s}$ over the surface of the quarter section.

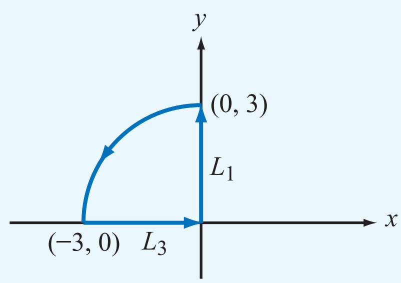

Figure P3.55 Problem 3.55.

Solution:

(a)

$$\oint_C \mathbf{B} \cdot d\boldsymbol{\ell} = \int_{L_1} \mathbf{B} \cdot d\boldsymbol{\ell} + \int_{L_2} \mathbf{B} \cdot d\boldsymbol{\ell} + \int_{L_3} \mathbf{B} \cdot d\boldsymbol{\ell}$$

Given the shape of the path, it is best to use cylindrical coordinates. $\mathbf{B}$ is already expressed in cylindrical coordinates, and we need to choose $d\boldsymbol{\ell}$ in cylindrical coordinates:

$$d\boldsymbol{\ell} = \hat{\mathbf{r}}\; dr + \hat{\boldsymbol{\phi}} r\; d\phi + \hat{\mathbf{z}}\; dz.$$

Along path L_1, $d\phi = 0$ and $dz = 0$. Hence, $d\boldsymbol{\ell} = \hat{\mathbf{r}}\; dr$ and

$$\int_{L_1} \mathbf{B} \cdot d\boldsymbol{\ell} = \int_{r=0}^{r=3} (\hat{\mathbf{r}} \cos\phi + \hat{\boldsymbol{\phi}} \sin\phi) \cdot \hat{\mathbf{r}}\; dr \bigg|_{\phi=90^\circ}$$

$$= \int_{r=0}^{3} \cos\phi\; dr \bigg|_{\phi=90^\circ} = r\cos\phi\big|_{r=0}^{3}\Big|_{\phi=90^\circ} = 0.$$

Along L_2, $dr = dz = 0$. Hence, $d\boldsymbol{\ell} = \hat{\boldsymbol{\phi}} r\; d\phi$ and

$$\int_{L_2} \mathbf{B} \cdot d\boldsymbol{\ell} = \int_{\phi=90^\circ}^{180^\circ} (\hat{\mathbf{r}} \cos\phi + \hat{\boldsymbol{\phi}} \sin\phi) \cdot \hat{\boldsymbol{\phi}} r\; d\phi \bigg|_{r=3}$$

$$= -3\cos\phi|_{90^\circ}^{180^\circ} = 3.$$

Along L_3, $dz = 0$ and $d\phi = 0$. Hence, $d\boldsymbol{\ell} = \hat{\mathbf{r}}\ dr$ and

$$\int_{L_3} \mathbf{B}\cdot d\boldsymbol{\ell} = \int_{r=3}^{0} (\hat{\mathbf{r}}\cos\phi + \hat{\boldsymbol{\phi}}\sin\phi)\cdot\hat{\mathbf{r}}\ dr\bigg|_{\phi=180^\circ}$$

$$= \int_{r=3}^{0} \cos\phi\ dr|_{\phi=180^\circ} = -r|_3^0 = 3.$$

Hence,

$$\oint_C \mathbf{B}\cdot d\boldsymbol{\ell} = 0+3+3 = 6.$$

(b)

$$\nabla\times\mathbf{B} = \hat{\mathbf{z}}\frac{1}{r}\left(\frac{\partial}{\partial r}\left(rB_\phi - \frac{\partial B_r}{\partial\phi}\right)\right)$$

$$= \hat{\mathbf{z}}\frac{1}{r}\left(\frac{\partial}{\partial r}(r\sin\phi) - \frac{\partial}{\partial\phi}(\cos\phi)\right)$$

$$= \hat{\mathbf{z}}\frac{1}{r}(\sin\phi+\sin\phi) = \hat{\mathbf{z}}\frac{2}{r}\sin\phi.$$

$$\int_S (\nabla\times\mathbf{B})\cdot d\mathbf{s} = \int_{r=0}^{3}\int_{\phi=90^\circ}^{180^\circ}\left(\hat{\mathbf{z}}\frac{2}{r}\sin\phi\right)\cdot\hat{\mathbf{z}}\,r\,dr\,d\phi$$

$$= -2r|_{r=0}^{3}\cos\phi\bigg|_{\phi=90^\circ}^{180^\circ} = 6.$$

Hence, Stokes's theorem is verified.