4.16 A line of charge with uniform density ρ_l extends between $z = -L/2$ and $z = L/2$ along the z-axis. Apply Coulomb's law to obtain an expression for the electric field at any point $P(r, \phi, 0)$ on the x–y plane. Show that your result reduces to the expression given by (4.33) as the length L is extended to infinity.

Solution:

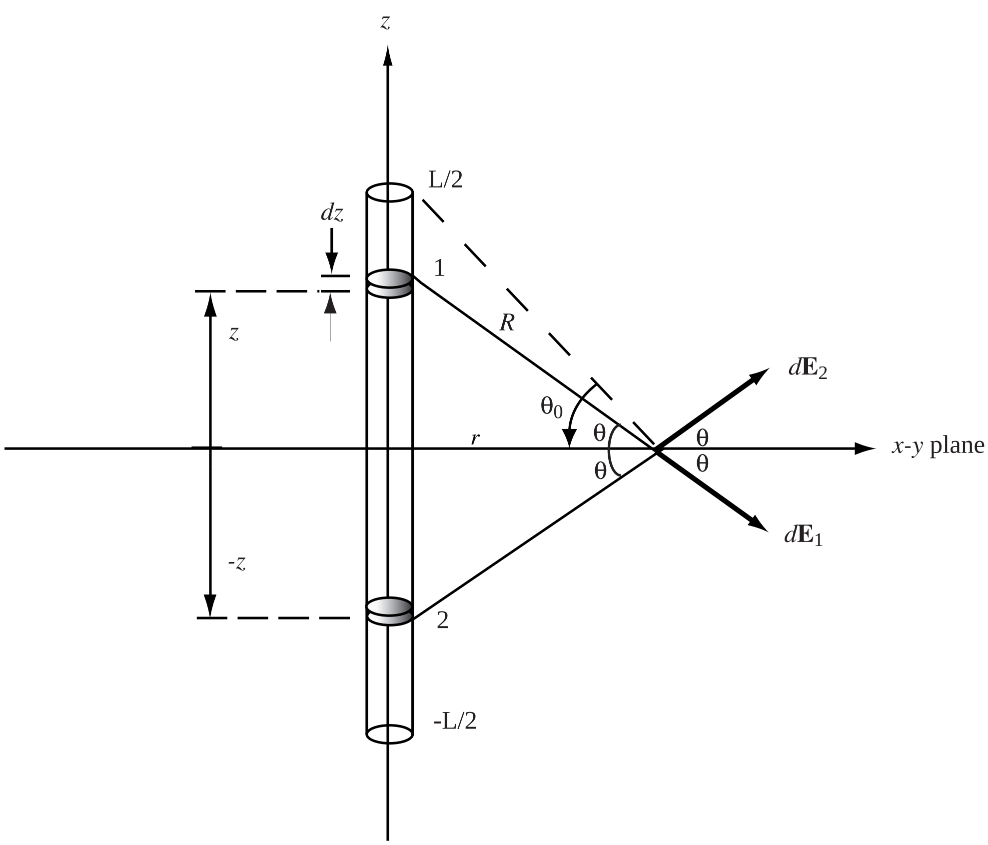

Figure P4.16 Line charge of length L.

Consider an element of charge of height dz at height z. Call it element 1. The electric field at P due to this element is $d\mathbf{E}_1$. Similarly, an element at $-z$ produces $d\mathbf{E}_2$. These two electric fields have equal z-components, but in opposite directions, and hence they will cancel. Their components along $\hat{\mathbf{r}}$ will add. Thus, the net field due to both elements is

$$d\mathbf{E} = d\mathbf{E}_1 + d\mathbf{E}_2 = \hat{\mathbf{r}}\frac{2\rho_l \cos\theta \; dz}{4\pi\varepsilon_0 R^2} = \frac{\hat{\mathbf{r}}\rho_l \cos\theta \; dz}{2\pi\varepsilon_0 R^2} \; .$$

where the $\cos\theta$ factor provides the components of $d\mathbf{E}_1$ and $d\mathbf{E}_2$ along $\hat{\mathbf{r}}$.

Our integration variable is z, but it will be easier to integrate over the variable θ from $\theta = 0$ to

$$\theta_0 = \sin^{-1} \frac{L/2}{\sqrt{r^2 + (L/2)^2}} \, .$$

Hence, with $R = r/\cos\theta$, and $z = r\tan\theta$ and $dz = r\sec^2\theta \, d\theta$, we have

$$\begin{aligned}
\mathbf{E} = \int_{z=0}^{L/2} d\mathbf{E} = \int_{\theta=0}^{\theta_0} d\mathbf{E} = \int_0^{\theta_0} \hat{\mathbf{r}} \frac{\rho_l}{2\pi\varepsilon_0} \frac{\cos^3\theta}{r^2} \, r\sec^2\theta \, d\theta \\
= \hat{\mathbf{r}} \frac{\rho_l}{2\pi\varepsilon_0 r} \int_0^{\theta_0} \cos\theta \, d\theta \\
= \hat{\mathbf{r}} \frac{\rho_l}{2\pi\varepsilon_0 r} \sin\theta_0 = \hat{\mathbf{r}} \frac{\rho_l}{2\pi\varepsilon_0 r} \frac{L/2}{\sqrt{r^2 + (L/2)^2}} \, .
\end{aligned}$$

For $L \gg r$,

$$\frac{L/2}{\sqrt{r^2 + (L/2)^2}} \approx 1,$$

and

$$\mathbf{E} = \hat{\mathbf{r}} \frac{\rho_l}{2\pi\varepsilon_0 r} \qquad \text{(infinite line of charge).}$$