4.32 A circular ring of charge of radius a lies in the x–y plane and is centered at the origin. Assume also that the ring is in air and carries a uniform density ρ_ℓ.

(a) Show that the electrical potential at $(0,0,z)$ is given by

$$V = \rho_\ell a/[2\varepsilon_0(a^2+z^2)^{1/2}].$$

(b) Find the corresponding electric field **E**.

Solution:

(a) For the ring of charge shown in Fig. P4.32, using Eq. (3.67) in Eq. (4.48c) gives

$$V\left(\vec{R}\right) = \frac{1}{4\pi\varepsilon_0}\int_{l'}\frac{\rho_l}{R'}\,dl' = \frac{1}{4\pi\varepsilon_0}\int_{\phi'=0}^{2\pi}\frac{\rho_l}{\sqrt{a^2+r^2-2ar\cos\left(\phi'-\phi\right)+z^2}}\,a\,d\phi'.$$

Point $(0,0,z)$ in Cartesian coordinates corresponds to $(r,\phi,z) = (0,\phi,z)$ in cylindrical coordinates. Hence, for $r = 0$,

$$V\left(0,0,z\right) = \frac{1}{4\pi\varepsilon_0}\int_{\phi'=0}^{2\pi}\frac{\rho_l}{\sqrt{a^2+z^2}}\,a\,d\phi' = \frac{\rho_l a}{2\varepsilon_0\sqrt{a^2+z^2}}.$$

(b) From Eq. (4.51),

$$\vec{E} = -\nabla V = -\hat{\mathbf{z}}\frac{\rho_l a}{2\varepsilon_0}\frac{\partial}{\partial z}\left(a^2+z^2\right)^{-1/2} = \hat{\mathbf{z}}\frac{\rho_l a}{2\varepsilon_0}\frac{z}{\left(a^2+z^2\right)^{3/2}} \qquad \text{(V/m)}.$$