4.5 Find the total charge on a circular disk defined by $r \le a$ and $z = 0$ if:

(a) $\rho_s = \rho_{s0} \cos\phi$ (C/m^2)

(b) $\rho_s = \rho_{s0} \sin^2\phi$ (C/m^2)

(c) $\rho_s = \rho_{s0} e^{-r}$ (C/m^2)

(d) $\rho_s = \rho_{s0} e^{-r} \sin^2\phi$ (C/m^2)

where ρ_{s0} is a constant.

Solution:

(a)

$$Q = \int \rho_s \, ds = \int_{r=0}^{a} \int_{\phi=0}^{2\pi} \rho_{s0} \cos\phi \;\; r\,dr\,d\phi = \rho_{s0} \left. \frac{r^2}{2} \right|_0^a \left. \sin\phi \right|_0^{2\pi} = 0.$$

(b)

$$\begin{aligned} Q = \int_{r=0}^{a} \int_{\phi=0}^{2\pi} \rho_{s0} \sin^2\phi \;\; r\,dr\,d\phi &= \rho_{s0} \left. \frac{r^2}{2} \right|_0^a \int_0^{2\pi} \left(\frac{1-\cos 2\phi}{2} \right) d\phi \\ &= \frac{\rho_{s0} a^2}{4} \left. \left(\phi - \frac{\sin 2\phi}{2} \right) \right|_0^{2\pi} = \frac{\pi a^2}{2} \rho_{s0}. \end{aligned}$$

(c)

$$\begin{aligned} Q = \int_{r=0}^{a} \int_{\phi=0}^{2\pi} \rho_{s0} e^{-r} r\,dr\,d\phi &= 2\pi\rho_{s0} \int_0^a r e^{-r}\,dr \\ &= 2\pi\rho_{s0} \left[-re^{-r} - e^{-r} \right]_0^a \\ &= 2\pi\rho_{s0} [1 - e^{-a}(1+a)]. \end{aligned}$$

(d)

$$\begin{aligned} Q &= \int_{r=0}^{a} \int_{\phi=0}^{2\pi} \rho_{s0} e^{-r} \sin^2\phi \;\; r\,dr\,d\phi \\ &= \rho_{s0} \int_{r=0}^{a} re^{-r}\,dr \int_{\phi=0}^{2\pi} \sin^2\phi\,d\phi \\ &= \rho_{s0} [1 - e^{-a}(1+a)] \cdot \pi = \pi\rho_{s0} [1 - e^{-a}(1+a)]. \end{aligned}$$