5.26 With reference to Fig. 5-10:

(a) Derive an expression for the vector magnetic potential **A** at a point P located at a distance r from the wire in the x–y plane.

(b) Derive **B** from **A**. Show that your result is identical with the expression given by Eq. (5.29), which was derived by applying the Biot–Savart law.

Solution:

(a) From the text immediately following Eq. (5.65), that equation may take the form

$$\begin{aligned}
\vec{A} = \frac{\mu}{4\pi}\int_{\ell'}\frac{I}{R'}d\vec{l}' &= \frac{\mu_0}{4\pi}\int_{z'=-\ell/2}^{\ell/2}\frac{I}{\sqrt{z'^2+r^2}}\,\hat{\mathbf{z}}\,dz' \\
&= \frac{\mu_0}{4\pi}\left(\hat{\mathbf{z}}I\ln\left(z'+\sqrt{z'^2+r^2}\right)\right)\Big|_{z'=-\ell/2}^{\ell/2} \\
&= \hat{\mathbf{z}}\frac{\mu_0 I}{4\pi}\ln\left(\frac{\ell/2+\sqrt{(\ell/2)^2+r^2}}{-\ell/2+\sqrt{(-\ell/2)^2+r^2}}\right) \\
&= \hat{\mathbf{z}}\frac{\mu_0 I}{4\pi}\ln\left(\frac{\ell+\sqrt{\ell^2+4r^2}}{-\ell+\sqrt{\ell^2+4r^2}}\right).
\end{aligned}$$

(b) From Eq. (5.53),

$$\begin{aligned}
\vec{B} &= \nabla\times\vec{A} \\
&= \nabla\times\left(\hat{\mathbf{z}}\frac{\mu_0 I}{4\pi}\ln\left(\frac{\ell+\sqrt{\ell^2+4r^2}}{-\ell+\sqrt{\ell^2+4r^2}}\right)\right) \\
&= -\hat{\boldsymbol{\phi}}\frac{\mu_0 I}{4\pi}\frac{\partial}{\partial r}\ln\left(\frac{\ell+\sqrt{\ell^2+4r^2}}{-\ell+\sqrt{\ell^2+4r^2}}\right) \\
&= -\hat{\boldsymbol{\phi}}\frac{\mu_0 I}{4\pi}\left(\frac{-\ell+\sqrt{\ell^2+4r^2}}{\ell+\sqrt{\ell^2+4r^2}}\right)\frac{\partial}{\partial r}\left(\frac{\ell+\sqrt{\ell^2+4r^2}}{-\ell+\sqrt{\ell^2+4r^2}}\right) \\
&= -\hat{\boldsymbol{\phi}}\frac{\mu_0 I}{4\pi}\left(\frac{-\ell+\sqrt{\ell^2+4r^2}}{\ell+\sqrt{\ell^2+4r^2}}\right) \\
&\times\left(\frac{\left(-\ell+\sqrt{\ell^2+4r^2}\right)\frac{\partial}{\partial r}\left(\ell+\sqrt{\ell^2+4r^2}\right)-\left(\ell+\sqrt{\ell^2+4r^2}\right)\frac{\partial}{\partial r}\left(-\ell+\sqrt{\ell^2+4r^2}\right)}{\left(-\ell+\sqrt{\ell^2+4r^2}\right)^2}\right)
\end{aligned}$$

$$= -\hat{\boldsymbol{\phi}}\frac{\mu_0 I}{4\pi}\left(\frac{\left(-\ell+\sqrt{\ell^2+4r^2}\right)-\left(\ell+\sqrt{\ell^2+4r^2}\right)}{\left(-\ell+\sqrt{\ell^2+4r^2}\right)\left(\ell+\sqrt{\ell^2+4r^2}\right)}\right)\frac{4r}{\sqrt{\ell^2+4r^2}}$$

$$= -\hat{\boldsymbol{\phi}}\frac{\mu_0 I}{4\pi}\left(\frac{-2\ell}{4r^2}\right)\frac{4r}{\sqrt{\ell^2+4r^2}} = \hat{\boldsymbol{\phi}}\frac{\mu_0 I\ell}{2\pi r\sqrt{\ell^2+4r^2}} \qquad \text{(T)}.$$

which is the same as Eq. (5.29).