6.20 If the current density in a conducting medium is given by

$$\mathbf{J}(x,y,z;t) = (\hat{\mathbf{x}}z - \hat{\mathbf{y}}4y^2 + \hat{\mathbf{z}}2x)\cos\omega t$$

determine the corresponding charge distribution $\rho_{\rm v}(x,y,z;t)$.

Solution: Eq. (6.58) is given by

$$\nabla\cdot\mathbf{J} = -\frac{\partial\rho_{\rm v}}{\partial t}\,. \tag{28}$$

The divergence of $\mathbf{J}$ is

$$\begin{aligned}\nabla\cdot\mathbf{J} &= \left(\hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right)\cdot(\hat{\mathbf{x}}z^2 - \hat{\mathbf{y}}4y^2 + \hat{\mathbf{z}}2x)\cos\omega t \\ &= -4\frac{\partial}{\partial y}(y^2\cos\omega t) = -8y\cos\omega t.\end{aligned}$$

Using this result in Eq. (28) and then integrating both sides with respect to t gives

$$\rho_{\rm v} = -\int(\nabla\cdot\mathbf{J})\;dt = -\int -8y\cos\omega t\;dt = \frac{8y}{\omega}\sin\omega t + C_0,$$

where C_0 is a constant of integration.