6.21 In a certain medium, the direction of current density $\mathbf{J}$ points in the radial direction in cylindrical coordinates and its magnitude is independent of both ϕ and z. Determine $\mathbf{J}$, given that the charge density in the medium is

$$\rho_\mathrm{v} = \rho_0 r \cos\omega t \quad (\mathrm{C/m^3}).$$

Solution: Based on the given information,

$$\mathbf{J} = \hat{\mathbf{r}} J_\mathrm{r}(r).$$

With $J_\phi = J_z = 0$, in cylindrical coordinates the divergence is given by

$$\nabla \cdot \mathbf{J} = \frac{1}{r}\frac{\partial}{\partial r}(rJ_\mathrm{r}).$$

From Eq. (6.54),

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_\mathrm{v}}{\partial t} = -\frac{\partial}{\partial t}(\rho_0 r \cos\omega t) = \rho_0 r \omega \sin\omega t.$$

Hence

$$\begin{aligned}
\frac{1}{r}\frac{\partial}{\partial r}(rJ_\mathrm{r}) &= \rho_0 r \omega \sin\omega t, \\
\frac{\partial}{\partial r}(rJ_\mathrm{r}) &= \rho_0 r^2 \omega \sin\omega t, \\
\int_0^r \frac{\partial}{\partial r}(rJ_\mathrm{r})\,dr &= \rho_0 \omega \sin\omega t \int_0^r r^2\,dr, \\
rJ_\mathrm{r}\big|_0^r &= (\rho_0 \omega \sin\omega t)\left.\frac{r^3}{3}\right|_0^r, \\
J_\mathrm{r} &= \frac{\rho_0 \omega r^2}{3}\sin\omega t,
\end{aligned}$$

and

$$\mathbf{J} = \hat{\mathbf{r}} J_\mathrm{r} = \hat{\mathbf{r}}\,\frac{\rho_0 \omega r^2}{3}\sin\omega t \qquad (\mathrm{A/m^2}).$$