3.46 The scalar function V is given by

$$V = \frac{2z}{x^2 + y^2} .$$

(a) Determine ∇V in Cartesian coordinates.

(b) Convert the result of part (a) from Cartesian to cylindrical coordinates.

(c) Convert the expression for V into cylindrical coordinates and then determine ∇V in those coordinates. Compare the results of parts (b) and (c).

Solution:

(a) In Cartesian coordinates

$$\begin{aligned}
\nabla V &= \hat{\mathbf{x}} \frac{\partial V}{\partial x} + \hat{\mathbf{y}} \frac{\partial V}{\partial y} + \hat{\mathbf{z}} \frac{\partial V}{\partial z} \\
&= \hat{\mathbf{x}} \frac{\partial}{\partial x} + \hat{\mathbf{y}} \frac{\partial}{\partial y} + \hat{\mathbf{z}} \frac{\partial}{\partial z} \left[2z(x^2 + y^2)^{-1} \right] \\
&= \frac{\hat{\mathbf{x}}(-4xz) + \hat{\mathbf{y}}(-4yz) + \hat{\mathbf{z}}\, 2(x^2 + y^2)}{(x^2 + y^2)^2} .
\end{aligned}$$

(b) To convert ∇V into cylindrical coordinates, we use the following relations from Table 3-2:

$$\begin{aligned}
x &= r\cos\phi, \\
y &= r\sin\phi, \\
z &= z, \\
\hat{\mathbf{x}} &= \hat{\mathbf{r}}\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi, \\
\hat{\mathbf{y}} &= \hat{\mathbf{r}}\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi, \\
\hat{\mathbf{z}} &= \hat{\mathbf{z}}.
\end{aligned}$$

The substitutions lead to

$$\begin{aligned}
\nabla V &= \frac{-\hat{\mathbf{r}}\, 4zr + \hat{\mathbf{z}}\, 2r^2}{r^4} \\
&= \frac{-\hat{\mathbf{r}}\, 4z + \hat{\mathbf{z}}\, 2r}{r^3} .
\end{aligned}$$

(c) In cylindrical coordinates,

$$\begin{aligned}
V &= \frac{2z}{x^2 + y^2} \\
&= \frac{2z}{r^2}
\end{aligned}$$

and

$$\begin{aligned}
\nabla V &= \hat{\mathbf{r}}\,\frac{\partial V}{\partial r} + \hat{\boldsymbol{\phi}}\,\frac{1}{r}\,\frac{\partial V}{\partial \phi} + \hat{\mathbf{z}}\,\frac{\partial V}{\partial z} \\
&= \hat{\mathbf{r}}\,\frac{\partial}{\partial r}\left(\frac{2z}{r^2}\right) + \hat{\boldsymbol{\phi}}\,\frac{1}{r}\,\frac{\partial}{\partial \phi}\left(\frac{2z}{r^2}\right) + \hat{\mathbf{z}}\,\frac{\partial}{\partial z}\left(\frac{2z}{r^2}\right) \\
&= -\hat{\mathbf{r}}\,\frac{4z}{r^3} + 0 + \hat{\mathbf{z}}\,\frac{2}{r^2} \\
&= \frac{-\hat{\mathbf{r}}\,4z + \hat{\mathbf{z}}\,2r}{r^3}\,,
\end{aligned}$$

which is identical with the result of part (b).