3.51 For the vector field $\mathbf{D} = \hat{\mathbf{R}}3R^2$, evaluate both sides of the divergence theorem for the region enclosed between the spherical shells defined by $R = 1$ and $R = 2$.

Solution: The divergence theorem is given by Eq. (3.98). Evaluating the left hand side:

$$\int_{\mathcal{V}} \nabla\cdot\vec{D}\, d\mathcal{V} = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{R=1}^{2}\left(\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\left(3R^2\right)\right)\right)R^2\sin\theta\, dR\, d\theta\, d\phi$$

$$= 2\pi\left(-\cos\theta\right)\big|_{\theta=0}^{\pi}\left(3R^4\right)\big|_{R=1}^{2} = 180\pi.$$

The right hand side evaluates to

$$\oint_S \vec{D}\cdot d\vec{s} = \left.\left(\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\left(\hat{\mathbf{R}}3R^2\right)\cdot\left(-\hat{\mathbf{R}}R^2\sin\theta\, d\theta\, d\phi\right)\right)\right|_{R=1}$$

$$+\left.\left(\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\left(\hat{\mathbf{R}}3R^2\right)\cdot\left(\hat{\mathbf{R}}R^2\sin\theta\, d\theta\, d\phi\right)\right)\right|_{R=2}$$

$$= -2\pi\int_{\theta=0}^{\pi}3\sin\theta\, d\theta + 2\pi\int_{\theta=0}^{\pi}48\sin\theta\, d\theta = 180\pi.$$