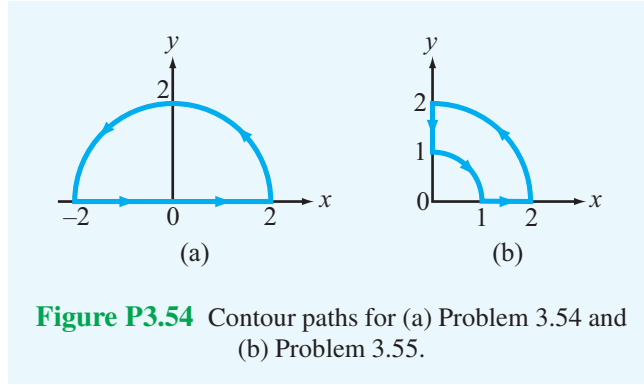


**3.54** Verify Stokes's theorem for the vector field  $\mathbf{B} = (\hat{\mathbf{r}}r \cos \phi + \hat{\boldsymbol{\phi}} \sin \phi)$  by evaluating:

- (a)  $\oint_C \mathbf{B} \cdot d\mathbf{l}$  over the semicircular contour shown in Fig. P3.54(a), and  
 (b)  $\int_S (\nabla \times \mathbf{B}) \cdot d\mathbf{s}$  over the surface of the semicircle.



**Figure P3.54** Contour paths for (a) Problem 3.54 and (b) Problem 3.55.

**Solution:**

(a)

$$\begin{aligned}
 \oint \vec{B} \cdot d\vec{l} &= \int_{L_1} \vec{B} \cdot d\vec{l} + \int_{L_2} \vec{B} \cdot d\vec{l} + \int_{L_3} \vec{B} \cdot d\vec{l}, \\
 \vec{B} \cdot d\vec{l} &= (\hat{\mathbf{r}}r \cos \phi + \hat{\boldsymbol{\phi}} \sin \phi) \cdot (\hat{\mathbf{r}} dr + \hat{\boldsymbol{\phi}} r d\phi + \hat{\mathbf{z}} dz) = r \cos \phi dr + r \sin \phi d\phi, \\
 \int_{L_1} \vec{B} \cdot d\vec{l} &= \left( \int_{r=0}^2 r \cos \phi dr \right) \Big|_{\phi=0, z=0} + \left( \int_{\phi=0}^0 r \sin \phi d\phi \right) \Big|_{z=0} \\
 &= \left( \frac{1}{2} r^2 \right) \Big|_{r=0}^2 + 0 = 2, \\
 \int_{L_2} \vec{B} \cdot d\vec{l} &= \left( \int_{r=2}^2 r \cos \phi dr \right) \Big|_{z=0} + \left( \int_{\phi=0}^{\pi} r \sin \phi d\phi \right) \Big|_{r=2, z=0} \\
 &= 0 + (-2 \cos \phi) \Big|_{\phi=0}^{\pi} = 4, \\
 \int_{L_3} \vec{B} \cdot d\vec{l} &= \left( \int_{r=2}^0 r \cos \phi dr \right) \Big|_{\phi=\pi, z=0} + \left( \int_{\phi=\pi}^{\pi} r \sin \phi d\phi \right) \Big|_{z=0} \\
 &= \left( -\frac{1}{2} r^2 \right) \Big|_{r=2}^0 + 0 = 2, \\
 \oint \vec{B} \cdot d\vec{l} &= 2 + 4 + 2 \\
 &= 8.
 \end{aligned}$$

(b)

$$\begin{aligned}\nabla \times \vec{B} &= \nabla \times (\hat{\mathbf{r}} r \cos \phi + \hat{\boldsymbol{\phi}} \sin \phi) \\&= \hat{\mathbf{r}} \left( \frac{1}{r} \frac{\partial}{\partial \phi} 0 - \frac{\partial}{\partial z} (\sin \phi) \right) + \hat{\boldsymbol{\phi}} \left( \frac{\partial}{\partial z} (r \cos \phi) - \frac{\partial}{\partial r} 0 \right) \\&\quad + \hat{\mathbf{z}} \frac{1}{r} \left( \frac{\partial}{\partial r} (r (\sin \phi)) - \frac{\partial}{\partial \phi} (r \cos \phi) \right) \\&= \hat{\mathbf{r}} 0 + \hat{\boldsymbol{\phi}} 0 + \hat{\mathbf{z}} \frac{1}{r} (\sin \phi + (r \sin \phi)) \\&= \hat{\mathbf{z}} \sin \phi \left( 1 + \frac{1}{r} \right), \\ \iint \nabla \times \vec{B} \cdot d\vec{s} &= \int_{\phi=0}^{\pi} \int_{r=0}^2 \left( \hat{\mathbf{z}} \sin \phi \left( 1 + \frac{1}{r} \right) \right) \cdot (\hat{\mathbf{z}} r dr d\phi) \\&= \int_{\phi=0}^{\pi} \int_{r=0}^2 \sin \phi (r+1) dr d\phi \\&= \left( (-\cos \phi (\tfrac{1}{2}r^2 + r)) \Big|_{r=0} \right) \Big|_{\phi=0}^{\pi} \\&= 8.\end{aligned}$$

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