

**3.55** Repeat Problem 3.54 for the contour shown in Fig. P3.54(b).

**Solution: (a)**

$$\begin{aligned}
 \oint \vec{B} \cdot d\vec{l} &= \int_{L_1} \vec{B} \cdot d\vec{l} + \int_{L_2} \vec{B} \cdot d\vec{l} + \int_{L_3} \vec{B} \cdot d\vec{l} + \int_{L_4} \vec{B} \cdot d\vec{l}, \\
 \vec{B} \cdot d\vec{l} &= (\hat{r}r \cos \phi + \hat{\phi} \sin \phi) \cdot (\hat{r} dr + \hat{\phi} r d\phi + \hat{z} dz) = r \cos \phi dr + r \sin \phi d\phi, \\
 \int_{L_1} \vec{B} \cdot d\vec{l} &= \left( \int_{r=1}^2 r \cos \phi dr \right) \Big|_{\phi=0, z=0} + \left( \int_{\phi=0}^0 r \sin \phi d\phi \right) \Big|_{z=0} \\
 &= \left( \frac{1}{2} r^2 \right) \Big|_{r=1}^2 + 0 = \frac{3}{2}, \\
 \int_{L_2} \vec{B} \cdot d\vec{l} &= \left( \int_{r=2}^1 r \cos \phi dr \right) \Big|_{z=0} + \left( \int_{\phi=0}^{\pi/2} r \sin \phi d\phi \right) \Big|_{r=2, z=0} \\
 &= 0 + (-2 \cos \phi) \Big|_{\phi=0}^{\pi/2} = 2, \\
 \int_{L_3} \vec{B} \cdot d\vec{l} &= \left( \int_{r=2}^1 r \cos \phi dr \right) \Big|_{\phi=\pi/2, z=0} + \left( \int_{\phi=\pi/2}^{\pi/2} r \sin \phi d\phi \right) \Big|_{z=0} = 0, \\
 \int_{L_4} \vec{B} \cdot d\vec{l} &= \left( \int_{r=1}^2 r \cos \phi dr \right) \Big|_{z=0} + \left( \int_{\phi=\pi/2}^0 r \sin \phi d\phi \right) \Big|_{r=1, z=0} \\
 &= 0 + (-\cos \phi) \Big|_{\phi=\pi/2}^0 = -1, \\
 \oint \vec{B} \cdot d\vec{l} &= \frac{3}{2} + 2 + 0 - 1 = \frac{5}{2}.
 \end{aligned}$$

**(b)**

$$\begin{aligned}
 \nabla \times \vec{B} &= \nabla \times (\hat{r}r \cos \phi + \hat{\phi} \sin \phi) \\
 &= \hat{r} \left( \frac{1}{r} \frac{\partial}{\partial \phi} 0 - \frac{\partial}{\partial z} (\sin \phi) \right) + \hat{\phi} \left( \frac{\partial}{\partial z} (r \cos \phi) - \frac{\partial}{\partial r} 0 \right) \\
 &\quad + \hat{z} \frac{1}{r} \left( \frac{\partial}{\partial r} (r (\sin \phi)) - \frac{\partial}{\partial \phi} (r \cos \phi) \right) \\
 &= \hat{r} 0 + \hat{\phi} 0 + \hat{z} \frac{1}{r} (\sin \phi + (r \sin \phi)) = \hat{z} \sin \phi \left( 1 + \frac{1}{r} \right), \\
 \iint \nabla \times \vec{B} \cdot d\vec{s} &= \int_{\phi=0}^{\pi/2} \int_{r=1}^2 \left( \hat{z} \sin \phi \left( 1 + \frac{1}{r} \right) \right) \cdot (\hat{z} r dr d\phi) \\
 &= \int_{\phi=0}^{\pi/2} \int_{r=1}^2 \sin \phi (r + 1) dr d\phi \\
 &= \left( (-\cos \phi \left( \frac{1}{2} r^2 + r \right)) \Big|_{r=1}^2 \right) \Big|_{\phi=0}^{\pi/2} = \frac{5}{2}.
 \end{aligned}$$


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