3.58 Determine if each of the following vector fields is solenoidal, conservative, or both:

(a) $\mathbf{A} = \hat{\mathbf{x}}x^2 - \hat{\mathbf{y}}y2xy$,
(b) $\mathbf{B} = \hat{\mathbf{x}}x^2 - \hat{\mathbf{y}}y^2 + \hat{\mathbf{z}}2z$,
(c) $\mathbf{C} = \hat{\mathbf{r}}(\sin\phi)/r^2 + \hat{\boldsymbol{\phi}}(\cos\phi)/r^2$,
(d) $\mathbf{D} = \hat{\mathbf{R}}/R$,
(e) $\mathbf{E} = \hat{\mathbf{r}}\left(3 - \frac{r}{1+r}\right) + \hat{\mathbf{z}}z$,
(f) $\mathbf{F} = (\hat{\mathbf{x}}y + \hat{\mathbf{y}}x)/(x^2 + y^2)$,
(g) $\mathbf{G} = \hat{\mathbf{x}}(x^2 + z^2) - \hat{\mathbf{y}}(y^2 + x^2) - \hat{\mathbf{z}}(y^2 + z^2)$,
(h) $\mathbf{H} = \hat{\mathbf{R}}(Re^{-R})$.

Solution:

(a)

$$\nabla\cdot\vec{A} = \nabla\cdot\left(\hat{\mathbf{x}}x^2 - \hat{\mathbf{y}}2xy\right) = \frac{\partial}{\partial x}x^2 - \frac{\partial}{\partial y}2xy = 2x - 2x = 0,$$

$$\begin{aligned}\nabla\times\vec{A} &= \nabla\times\left(\hat{\mathbf{x}}x^2 - \hat{\mathbf{y}}2xy\right)\\ &= \hat{\mathbf{x}}\left(\frac{\partial}{\partial y}0 - \frac{\partial}{\partial z}(-2xy)\right) + \hat{\mathbf{y}}\left(\frac{\partial}{\partial z}(x^2) - \frac{\partial}{\partial x}0\right) + \hat{\mathbf{z}}\left(\frac{\partial}{\partial x}(-2xy) - \frac{\partial}{\partial y}(x^2)\right)\\ &= \hat{\mathbf{x}}0 + \hat{\mathbf{y}}0 - \hat{\mathbf{z}}(2y) \neq 0.\end{aligned}$$

The field $\vec{A}$ is solenoidal but not conservative.

(b)

$$\nabla\cdot\vec{B} = \nabla\cdot\left(\hat{\mathbf{x}}x^2 - \hat{\mathbf{y}}y^2 + \hat{\mathbf{z}}2z\right) = \frac{\partial}{\partial x}x^2 - \frac{\partial}{\partial y}y^2 + \frac{\partial}{\partial z}2z = 2x - 2y + 2 \neq 0,$$

$$\begin{aligned}\nabla\times\vec{B} &= \nabla\times\left(\hat{\mathbf{x}}x^2 - \hat{\mathbf{y}}y^2 + \hat{\mathbf{z}}2z\right)\\ &= \hat{\mathbf{x}}\left(\frac{\partial}{\partial y}(2z) - \frac{\partial}{\partial z}(-y^2)\right) + \hat{\mathbf{y}}\left(\frac{\partial}{\partial z}(x^2) - \frac{\partial}{\partial x}(2z)\right)\\ &\quad + \hat{\mathbf{z}}\left(\frac{\partial}{\partial x}(-y^2) - \frac{\partial}{\partial y}(x^2)\right)\\ &= \hat{\mathbf{x}}0 + \hat{\mathbf{y}}0 + \hat{\mathbf{z}}0.\end{aligned}$$

The field $\vec{B}$ is conservative but not solenoidal.

(c)

$$\begin{aligned}\nabla\cdot\vec{C} &= \nabla\cdot\left(\hat{\mathbf{r}}\frac{\sin\phi}{r^2} + \hat{\boldsymbol{\phi}}\frac{\cos\phi}{r^2}\right)\\ &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\left(\frac{\sin\phi}{r^2}\right)\right) + \frac{1}{r}\frac{\partial}{\partial\phi}\left(\frac{\cos\phi}{r^2}\right) + \frac{\partial}{\partial z}0\\ &= \frac{-\sin\phi}{r^3} + \frac{-\sin\phi}{r^3} + 0 = \frac{-2\sin\phi}{r^3},\end{aligned}$$

$$
\begin{aligned}
\nabla\times\vec{C} &= \nabla\times\left(\hat{\mathbf{r}}\frac{\sin\phi}{r^2}+\hat{\boldsymbol{\phi}}\frac{\cos\phi}{r^2}\right) \\
&= \hat{\mathbf{r}}\left(\frac{1}{r}\frac{\partial}{\partial\phi}0-\frac{\partial}{\partial z}\left(\frac{\cos\phi}{r^2}\right)\right)+\hat{\boldsymbol{\phi}}\left(\frac{\partial}{\partial z}\left(\frac{\sin\phi}{r^2}\right)-\frac{\partial}{\partial r}0\right) \\
&\quad +\hat{\mathbf{z}}\frac{1}{r}\left(\frac{\partial}{\partial r}\left(r\left(\frac{\cos\phi}{r^2}\right)\right)-\frac{\partial}{\partial\phi}\left(\frac{\sin\phi}{r^2}\right)\right) \\
&= \hat{\mathbf{r}}0+\hat{\boldsymbol{\phi}}0+\hat{\mathbf{z}}\frac{1}{r}\left(-\left(\frac{\cos\phi}{r^2}\right)-\left(\frac{\cos\phi}{r^2}\right)\right)=\hat{\mathbf{z}}\frac{-2\cos\phi}{r^3}.
\end{aligned}
$$

The field $\vec{C}$ is neither solenoidal nor conservative.

(d)

$$
\nabla\cdot\vec{D}=\nabla\cdot\left(\frac{\hat{\mathbf{R}}}{R}\right)=\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\left(\frac{1}{R}\right)\right)+\frac{1}{R\sin\theta}\frac{\partial}{\partial\theta}(0\sin\theta)+\frac{1}{R\sin\theta}\frac{\partial}{\partial\phi}0=\frac{1}{R^2},
$$

$$
\begin{aligned}
\nabla\times\vec{D} &= \nabla\times\left(\frac{\hat{\mathbf{R}}}{R}\right) \\
&= \hat{\mathbf{R}}\frac{1}{R\sin\theta}\left(\frac{\partial}{\partial\theta}(0\sin\theta)-\frac{\partial}{\partial\phi}0\right)+\hat{\boldsymbol{\theta}}\frac{1}{R}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{1}{R}\right)-\frac{\partial}{\partial R}(R(0))\right) \\
&\quad +\hat{\boldsymbol{\phi}}\frac{1}{R}\left(\frac{\partial}{\partial R}(R(0))-\frac{\partial}{\partial\theta}\left(\frac{1}{R}\right)\right)=\hat{\mathbf{r}}0+\hat{\boldsymbol{\theta}}0+\hat{\boldsymbol{\phi}}0.
\end{aligned}
$$

The field $\vec{D}$ is conservative but not solenoidal.

(e)

$$
\begin{aligned}
\mathbf{E} &= \hat{\mathbf{r}}\left(3-\frac{r}{1+r}\right)+\hat{\mathbf{z}}z, \\
\nabla\cdot\mathbf{E} &= \frac{1}{r}\frac{\partial}{\partial r}(rE_{\mathrm{r}})+\frac{1}{r}\frac{\partial E_\phi}{\partial\phi}+\frac{\partial E_z}{\partial z} \\
&= \frac{1}{r}\frac{\partial}{\partial r}\left(3r-\frac{r^2}{1+r}\right)+1 \\
&= \frac{1}{r}\left[3-\frac{2r}{1+r}+\frac{r^2}{(1+r)^2}\right]+1 \\
&= \frac{1}{r}\left[\frac{3+3r^2+6r-2r-2r^2+r^2}{(1+r)^2}\right]+1=\frac{2r^2+4r+3}{r(1+r)^2}+1\neq 0,
\end{aligned}
$$

$$
\nabla\times\mathbf{E}=\hat{\mathbf{r}}\left(\frac{1}{r}\frac{\partial E_z}{\partial\phi}-\frac{\partial E_\phi}{\partial z}\right)+\hat{\boldsymbol{\phi}}\left(\frac{\partial E_r}{\partial z}-\frac{\partial E_z}{\partial r}\right)+\hat{\mathbf{z}}\left(\frac{1}{r}\frac{\partial}{\partial r}(rE_\phi)-\frac{1}{r}\frac{\partial E_r}{\partial\phi}\right)=0.
$$

Hence, $\mathbf{E}$ is conservative, but not solenoidal.

(f)

$$
\mathbf{F}=\frac{\hat{\mathbf{x}}y+\hat{\mathbf{y}}x}{x^2+y^2}=\hat{\mathbf{x}}\frac{y}{x^2+y^2}+\hat{\mathbf{y}}\frac{x}{x^2+y^2},
$$

$$
\begin{aligned}
\nabla \cdot \mathbf{F} &= \frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right) \\
&= \frac{-2xy}{(x^2+y^2)^2} + \frac{-2xy}{(x^2+y^2)^2} \neq 0, \\
\nabla \times \mathbf{F} &= \hat{\mathbf{x}}(0-0) + \hat{\mathbf{y}}(0-0) + \hat{\mathbf{z}}\left[\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)\right] \\
&= \hat{\mathbf{z}}\left(\frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2} - \frac{1}{x^2+y^2} + \frac{2y^2}{(x^2+y^2)^2}\right) \\
&= \hat{\mathbf{z}}\,\frac{2(y^2-x^2)}{(x^2+y^2)^2} \neq 0.
\end{aligned}
$$

Hence, $\mathbf{F}$ is neither solenoidal nor conservative.

(g)

$$
\begin{aligned}
\mathbf{G} &= \hat{\mathbf{x}}(x^2+z^2) - \hat{\mathbf{y}}(y^2+x^2) - \hat{\mathbf{z}}(y^2+z^2), \\
\nabla \cdot \mathbf{G} &= \frac{\partial}{\partial x}(x^2+z^2) - \frac{\partial}{\partial y}(y^2+x^2) - \frac{\partial}{\partial z}(y^2+z^2) \\
&= 2x - 2y - 2z \neq 0, \\
\nabla \times \mathbf{G} &= \hat{\mathbf{x}}\left(-\frac{\partial}{\partial y}(y^2+z^2) + \frac{\partial}{\partial z}(y^2+x^2)\right) + \hat{\mathbf{y}}\left(\frac{\partial}{\partial z}(x^2+z^2) + \frac{\partial}{\partial x}(y^2+z^2)\right) \\
&\quad + \hat{\mathbf{z}}\left(-\frac{\partial}{\partial x}(y^2+x^2) - \frac{\partial}{\partial y}(x^2+z^2)\right) \\
&= -\hat{\mathbf{x}}\,2y + \hat{\mathbf{y}}\,2z - \hat{\mathbf{z}}\,2x \neq 0.
\end{aligned}
$$

Hence, $\mathbf{G}$ is neither solenoidal nor conservative.

(h)

$$
\begin{aligned}
\mathbf{H} &= \hat{\mathbf{R}}(Re^{-R}), \\
\nabla \cdot \mathbf{H} &= \frac{1}{R^2}\frac{\partial}{\partial R}(R^3 e^{-R}) = \frac{1}{R^2}(3R^2 e^{-R} - R^3 e^{-R}) = e^{-R}(3-R) \neq 0, \\
\nabla \times \mathbf{H} &= 0.
\end{aligned}
$$

Hence, $\mathbf{H}$ is conservative, but not solenoidal.