

3.8 By expansion in Cartesian coordinates, prove:

- (a) the relation for the scalar triple product given by Eq. (3.29), and
- (b) the relation for the vector triple product given by Eq. (3.33).

Solution:

- (a) Proof of the scalar triple product given by Eq. (3.29): From Eq. (3.27),

$$\begin{aligned}\vec{A} \times \vec{B} &= \hat{x}(A_y B_z - A_z B_y) + \hat{y}(A_z B_x - A_x B_z) + \hat{z}(A_x B_y - A_y B_x), \\ \vec{B} \times \vec{C} &= \hat{x}(B_y C_z - B_z C_y) + \hat{y}(B_z C_x - B_x C_z) + \hat{z}(B_x C_y - B_y C_x), \\ \vec{C} \times \vec{A} &= \hat{x}(C_y A_z - C_z A_y) + \hat{y}(C_z A_x - C_x A_z) + \hat{z}(C_x A_y - C_y A_x).\end{aligned}$$

Employing Eq. (3.21), it is easily shown that

$$\begin{aligned}\vec{A} \cdot (\vec{B} \times \vec{C}) &= A_x(B_y C_z - B_z C_y) + A_y(B_z C_x - B_x C_z) + A_z(B_x C_y - B_y C_x), \\ \vec{B} \cdot (\vec{C} \times \vec{A}) &= B_x(C_y A_z - C_z A_y) + B_y(C_z A_x - C_x A_z) + B_z(C_x A_y - C_y A_x), \\ \vec{C} \cdot (\vec{A} \times \vec{B}) &= C_x(A_y B_z - A_z B_y) + C_y(A_z B_x - A_x B_z) + C_z(A_x B_y - A_y B_x),\end{aligned}$$

which are all the same.

- (b) Proof of the vector triple product given by Eq. (3.33): The evaluation of the left hand side employs the expression above for $\vec{B} \times \vec{C}$ with Eq. (3.27):

$$\begin{aligned}\vec{A} \times (\vec{B} \times \vec{C}) &= \vec{A} \times (\hat{x}(B_y C_z - B_z C_y) + \hat{y}(B_z C_x - B_x C_z) + \hat{z}(B_x C_y - B_y C_x)) \\ &= \hat{x}(A_y(B_x C_y - B_y C_x) - A_z(B_z C_x - B_x C_z)) \\ &\quad + \hat{y}(A_z(B_y C_z - B_z C_y) - A_x(B_x C_y - B_y C_x)) \\ &\quad + \hat{z}(A_x(B_z C_x - B_x C_z) - A_y(B_y C_z - B_z C_y)),\end{aligned}$$

while the right hand side, evaluated with the aid of Eq. (3.21), is

$$\begin{aligned}\vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B}) &= \vec{B}(A_x C_x + A_y C_y + A_z C_z) - \vec{C}(A_x B_x + A_y B_y + A_z B_z) \\ &= \hat{x}(B_x(A_y C_y + A_z C_z) - C_x(A_y B_y + A_z B_z)) \\ &\quad + \hat{y}(B_y(A_x C_x + A_z C_z) - C_y(A_x B_x + A_z B_z)) \\ &\quad + \hat{z}(B_z(A_x C_x + A_y C_y) - C_z(A_x B_x + A_y B_y)).\end{aligned}$$

By rearranging the expressions for the components, the left hand side is equal to the right hand side.
