7.41 Given a wave with

$$\mathbf{E} = \hat{\mathbf{x}} E_0 \cos(\omega t - kz)$$

calculate:

(a) The time-average electric energy density

$$(w_{\text{e}})_{\text{av}} = \frac{1}{T}\int_0^T w_{\text{e}}\,dt = \frac{1}{2T}\int_0^T \varepsilon E^2\,dt$$

(b) The time-average magnetic energy density

$$(w_{\text{m}})_{\text{av}} = \frac{1}{T}\int_0^T w_{\text{m}}\,dt = \frac{1}{2T}\int_0^T \mu H^2\,dt$$

(c) Show that $(w_{\text{e}})_{\text{av}} = (w_{\text{m}})_{\text{av}}$.

Solution:

(a)

$$(w_{\text{e}})_{\text{av}} = \frac{1}{2T}\int_0^T \varepsilon E_0^2 \cos^2(\omega t - kz)\,dt.$$

With $T = \frac{2\pi}{\omega}$,

$$\begin{aligned}
(w_{\text{e}})_{\text{av}} &= \frac{\omega \varepsilon E_0^2}{4\pi}\int_0^{2\pi/\omega} \cos^2(\omega t - kz)\,dt \\
&= \frac{\varepsilon E_0^2}{4\pi}\int_0^{2\pi} \cos^2(\omega t - kz)\,d(\omega t) \\
&= \frac{\varepsilon E_0^2}{4}.
\end{aligned}$$

(b)

$$\mathbf{H} = \hat{\mathbf{y}}\frac{E_0}{\eta}\cos(\omega t - kz).$$

$$\begin{aligned}
(w_{\text{m}})_{\text{av}} &= \frac{1}{2T}\int_0^T \mu H^2\,dt \\
&= \frac{1}{2T}\int_0^T \mu\frac{E_0^2}{\eta^2}\cos^2(\omega t - kz)\,dt \\
&= \frac{\mu E_0^2}{4\eta^2}.
\end{aligned}$$

(c)

$$(w_{\text{m}})_{\text{av}} = \frac{\mu E_0^2}{4\eta^2} = \frac{\mu E_0^2}{4\left(\frac{\mu}{\varepsilon}\right)} = \frac{\varepsilon E_0^2}{4} = (w_{\text{e}})_{\text{av}}.$$