8.2 A plane wave traveling in medium 1 with $\varepsilon_{r1} = 2.25$ is normally incident upon medium 2 with $\varepsilon_{r2} = 4$. Both media are made of nonmagnetic, non-conducting materials. If the electric field of the incident wave is given by

$$\mathbf{E}^{\mathrm{i}} = \hat{\mathbf{y}} 8\cos(6\pi \times 10^9 t - 30\pi x) \qquad \text{(V/m)}.$$

(a) Obtain time-domain expressions for the electric and magnetic fields in each of the two media.

(b) Determine the average power densities of the incident, reflected and transmitted waves.

Solution:

(a)

$$\mathbf{E}^{\mathrm{i}} = \hat{\mathbf{y}}\, 8\cos(6\pi \times 10^9 t - 30\pi x) \qquad \text{(V/m)},$$

$$\eta_1 = \frac{\eta_0}{\sqrt{\varepsilon_{r_1}}} = \frac{\eta_0}{\sqrt{2.25}} = \frac{\eta_0}{1.5} = \frac{377}{1.5} = 251.33\ \Omega,$$

$$\eta_2 = \frac{\eta_0}{\sqrt{\varepsilon_{r_2}}} = \frac{\eta_0}{\sqrt{4}} = \frac{377}{2} = 188.5\ \Omega,$$

$$\Gamma = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = \frac{1/2 - 1/1.5}{1/2 + 1/1.5} = -0.143,$$

$$\tau = 1 + \Gamma = 1 - 0.143 = 0.857,$$

$$\mathbf{E}^{\mathrm{r}} = \Gamma\mathbf{E}^{\mathrm{i}} = -1.14\,\hat{\mathbf{y}}\cos(6\pi \times 10^9 t + 30\pi x) \qquad \text{(V/m)}.$$

Note that the coefficient of x is positive, denoting the fact that $\mathbf{E}^{\mathrm{r}}$ belongs to a wave traveling in $-x$-direction.

$$\mathbf{E}_1 = \mathbf{E}^{\mathrm{i}} + \mathbf{E}^{\mathrm{r}} = \hat{\mathbf{y}}\,[8\cos(6\pi \times 10^9 t - 30\pi x) - 1.14\cos(6\pi \times 10^9 t + 30\pi x)] \qquad \text{(A/m)},$$

$$\mathbf{H}^{\mathrm{i}} = \hat{\mathbf{z}}\frac{8}{\eta_1}\cos(6\pi \times 10^9 t - 30\pi x) = \hat{\mathbf{z}}\,31.83\cos(6\pi \times 10^9 t - 30\pi x) \qquad \text{(mA/m)},$$

$$\mathbf{H}^{\mathrm{r}} = \hat{\mathbf{z}}\frac{1.14}{\eta_1}\cos(6\pi \times 10^9 t + 30\pi x) = \hat{\mathbf{z}}\,4.54\cos(6\pi \times 10^9 t + 30\pi x) \qquad \text{(mA/m)},$$

$$\begin{aligned}\mathbf{H}_1 &= \mathbf{H}^{\mathrm{i}} + \mathbf{H}^{\mathrm{r}} \\ &= \hat{\mathbf{z}}\,[31.83\cos(6\pi \times 10^9 t - 30\pi x) + 4.54\cos(6\pi \times 10^9 t + 30\pi x)] \qquad \text{(mA/m)}.\end{aligned}$$

Since $k_1 = \omega\sqrt{\mu\varepsilon_1}$ and $k_2 = \omega\sqrt{\mu\varepsilon_2}$,

$$k_2 = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\,k_1 = \sqrt{\frac{4}{2.25}}\,30\pi = 40\pi \qquad \text{(rad/m)},$$

$$\mathbf{E}_2 = \mathbf{E}^{\mathrm{t}} = \hat{\mathbf{y}}\,8\tau\cos(6\pi \times 10^9 t - 40\pi x) = \hat{\mathbf{y}}\,6.86\cos(6\pi \times 10^9 t - 40\pi x) \qquad \text{(V/m)},$$

$$\mathbf{H}_2 = \mathbf{H}^{\mathrm{t}} = \hat{\mathbf{z}}\frac{8\tau}{\eta_2}\cos(6\pi \times 10^9 t - 40\pi x) = \hat{\mathbf{z}}\,36.38\cos(6\pi \times 10^9 t - 40\pi x) \qquad \text{(mA/m)}.$$

(b)

$$\mathbf{S}^{\mathrm{i}}_{\mathrm{av}} = \hat{\mathbf{x}}\,\frac{8^2}{2\eta_1} = \frac{64}{2 \times 251.33} = \hat{\mathbf{x}}\,127.3 \qquad (\mathrm{mW/m^2}),$$

$$\mathbf{S}^{\mathrm{r}}_{\mathrm{av}} = -|\Gamma|^2 \mathbf{S}^{\mathrm{i}}_{\mathrm{av}} = -\hat{\mathbf{x}}\,(0.143)^2 \times 0.127 = -\hat{\mathbf{x}}\,2.6 \qquad (\mathrm{mW/m^2}),$$

$$\begin{aligned} \mathbf{S}^{\mathrm{t}}_{\mathrm{av}} &= \frac{|E^{\mathrm{t}}_0|^2}{2\eta_2} \\ &= \hat{\mathbf{x}}\,\tau^2 \frac{(8)^2}{2\eta_2} = \hat{\mathbf{x}}\,\frac{(0.86)^2 64}{2 \times 188.5} = \hat{\mathbf{x}}\,124.7 \qquad (\mathrm{mW/m^2}). \end{aligned}$$

Within calculation error, $\mathbf{S}^{\mathrm{i}}_{\mathrm{av}} + \mathbf{S}^{\mathrm{r}}_{\mathrm{av}} = \mathbf{S}^{\mathrm{t}}_{\mathrm{av}}$.