

Problem 3.15 A certain plane is described by

$$2x + 3y + 4z = 16.$$

Find the unit vector normal to the surface in the direction away from the origin.

Solution: Procedure:

1. Use the equation for the given plane to find three points, P_1 , P_2 and P_3 on the plane.
2. Find vector \mathbf{A} from P_1 to P_2 and vector \mathbf{B} from P_1 to P_3 .
3. Cross product of \mathbf{A} and \mathbf{B} gives a vector \mathbf{C} orthogonal to \mathbf{A} and \mathbf{B} , and hence to the plane.
4. Check direction of $\hat{\mathbf{c}}$.

Steps:

1. Choose the following three points:

$$P_1 \text{ at } (0, 0, 4),$$

$$P_2 \text{ at } (8, 0, 0),$$

$$P_3 \text{ at } (0, \frac{16}{3}, 0).$$

2. Vector \mathbf{A} from P_1 to P_2

$$\mathbf{A} = \hat{\mathbf{x}}(8 - 0) + \hat{\mathbf{y}}(0 - 0) + \hat{\mathbf{z}}(0 - 4) = \hat{\mathbf{x}}8 - \hat{\mathbf{z}}4$$

Vector \mathbf{B} from P_1 to P_3

$$\mathbf{B} = \hat{\mathbf{x}}(0 - 0) + \hat{\mathbf{y}}\left(\frac{16}{3} - 0\right) + \hat{\mathbf{z}}(0 - 4) = \hat{\mathbf{y}}\frac{16}{3} - \hat{\mathbf{z}}4$$

- 3.

$$\mathbf{C} = \mathbf{A} \times \mathbf{B}$$

$$= \hat{\mathbf{x}}(A_y B_z - A_z B_y) + \hat{\mathbf{y}}(A_z B_x - A_x B_z) + \hat{\mathbf{z}}(A_x B_y - A_y B_x)$$

$$= \hat{\mathbf{x}}\left(0 \cdot (-4) - (-4) \cdot \frac{16}{3}\right) + \hat{\mathbf{y}}((-4) \cdot 0 - 8 \cdot (-4)) + \hat{\mathbf{z}}\left(8 \cdot \frac{16}{3} - 0 \cdot 0\right)$$

$$= \hat{\mathbf{x}}\frac{64}{3} + \hat{\mathbf{y}}32 + \hat{\mathbf{z}}\frac{128}{3}$$

Verify that **C** is orthogonal to **A** and **B**

$$\mathbf{A} \cdot \mathbf{C} = \left(8 \cdot \frac{64}{3}\right) + (32 \cdot 0) + \left(\frac{128}{3} \cdot (-4)\right) = \frac{512}{3} - \frac{512}{3} = 0$$

$$\mathbf{B} \cdot \mathbf{C} = \left(0 \cdot \frac{64}{3}\right) + \left(32 \cdot \frac{16}{3}\right) + \left(\frac{128}{3} \cdot (-4)\right) = \frac{512}{3} - \frac{512}{3} = 0$$

4. $\mathbf{C} = \hat{\mathbf{x}} \frac{64}{3} + \hat{\mathbf{y}} 32 + \hat{\mathbf{z}} \frac{128}{3}$

$$\hat{\mathbf{c}} = \frac{\mathbf{C}}{|\mathbf{C}|} = \frac{\hat{\mathbf{x}} \frac{64}{3} + \hat{\mathbf{y}} 32 + \hat{\mathbf{z}} \frac{128}{3}}{\sqrt{\left(\frac{64}{3}\right)^2 + 32^2 + \left(\frac{128}{3}\right)^2}} = \hat{\mathbf{x}} 0.37 + \hat{\mathbf{y}} 0.56 + \hat{\mathbf{z}} 0.74.$$

$\hat{\mathbf{c}}$ points away from the origin as desired.
