

Problem 3.35 Transform the following vectors into spherical coordinates and then evaluate them at the indicated points:

- (a) $\mathbf{A} = \hat{\mathbf{x}}y^2 + \hat{\mathbf{y}}xz + \hat{\mathbf{z}}4$ at $P_1 = (1, -1, 2)$,
 (b) $\mathbf{B} = \hat{\mathbf{y}}(x^2 + y^2 + z^2) - \hat{\mathbf{z}}(x^2 + y^2)$ at $P_2 = (-1, 0, 2)$,
 (c) $\mathbf{C} = \hat{\mathbf{r}}\cos\phi - \hat{\phi}\sin\phi + \hat{\mathbf{z}}\cos\phi\sin\phi$ at $P_3 = (2, \pi/4, 2)$, and
 (d) $\mathbf{D} = \hat{\mathbf{x}}y^2/(x^2 + y^2) - \hat{\mathbf{y}}x^2/(x^2 + y^2) + \hat{\mathbf{z}}4$ at $P_4 = (1, -1, 2)$.

Solution: From Table 3-2:

(a)

$$\begin{aligned}\mathbf{A} &= (\hat{\mathbf{R}}\sin\theta\cos\phi + \hat{\boldsymbol{\theta}}\cos\theta\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi)(R\sin\theta\sin\phi)^2 \\ &\quad + (\hat{\mathbf{R}}\sin\theta\sin\phi + \hat{\boldsymbol{\theta}}\cos\theta\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi)(R\sin\theta\cos\phi)(R\cos\theta) \\ &\quad + (\hat{\mathbf{R}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta)4 \\ &= \hat{\mathbf{R}}(R^2\sin^2\theta\sin\phi\cos\phi(\sin\theta\sin\phi + \cos\theta) + 4\cos\theta) \\ &\quad + \hat{\boldsymbol{\theta}}(R^2\sin\theta\cos\theta\sin\phi\cos\phi(\sin\theta\sin\phi + \cos\theta) - 4\sin\theta) \\ &\quad + \hat{\boldsymbol{\phi}}R^2\sin\theta(\cos\theta\cos^2\phi - \sin\theta\sin^3\phi), \\ P_1 &= \left(\sqrt{1^2 + (-1)^2 + 2^2}, \tan^{-1} \left(\sqrt{1^2 + (-1)^2}/2 \right), \tan^{-1}(-1/1) \right) \\ &= (\sqrt{6}, 35.3^\circ, -45^\circ), \\ \mathbf{A}(P_1) &\approx \hat{\mathbf{R}}2.856 - \hat{\boldsymbol{\theta}}2.888 + \hat{\boldsymbol{\phi}}2.123.\end{aligned}$$

(b)

$$\begin{aligned}\mathbf{B} &= (\hat{\mathbf{R}}\sin\theta\sin\phi + \hat{\boldsymbol{\theta}}\cos\theta\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi)R^2 - (\hat{\mathbf{R}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta)R^2\sin^2\theta \\ &= \hat{\mathbf{R}}R^2\sin\theta(\sin\phi - \sin\theta\cos\theta) + \hat{\boldsymbol{\theta}}R^2(\cos\theta\sin\phi + \sin^3\theta) + \hat{\boldsymbol{\phi}}R^2\cos\phi, \\ P_2 &= \left(\sqrt{(-1)^2 + 0^2 + 2^2}, \tan^{-1} \left(\sqrt{(-1)^2 + 0^2}/2 \right), \tan^{-1}(0/(-1)) \right) \\ &= (\sqrt{5}, 26.6^\circ, 180^\circ), \\ \mathbf{B}(P_2) &\approx -\hat{\mathbf{R}}0.896 + \hat{\boldsymbol{\theta}}0.449 - \hat{\boldsymbol{\phi}}5.\end{aligned}$$

(c)

$$\begin{aligned}\mathbf{C} &= (\hat{\mathbf{R}}\sin\theta + \hat{\boldsymbol{\theta}}\cos\theta)\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi + (\hat{\mathbf{R}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta)\cos\phi\sin\phi \\ &= \hat{\mathbf{R}}\cos\phi(\sin\theta + \cos\theta\sin\phi) + \hat{\boldsymbol{\theta}}\cos\phi(\cos\theta - \sin\theta\sin\phi) - \hat{\boldsymbol{\phi}}\sin\phi, \\ P_3 &= \left(\sqrt{2^2 + 2^2}, \tan^{-1}(2/2), \pi/4 \right) = (2\sqrt{2}, 45^\circ, 45^\circ), \\ \mathbf{C}(P_3) &\approx \hat{\mathbf{R}}0.854 + \hat{\boldsymbol{\theta}}0.146 - \hat{\boldsymbol{\phi}}0.707.\end{aligned}$$

(d)

$$\begin{aligned}\mathbf{D} &= (\hat{\mathbf{R}} \sin \theta \cos \phi + \hat{\boldsymbol{\theta}} \cos \theta \cos \phi - \hat{\boldsymbol{\phi}} \sin \phi) \frac{R^2 \sin^2 \theta \sin^2 \phi}{R^2 \sin^2 \theta \sin^2 \phi + R^2 \sin^2 \theta \cos^2 \phi} \\ &\quad - (\hat{\mathbf{R}} \sin \theta \sin \phi + \hat{\boldsymbol{\theta}} \cos \theta \sin \phi + \hat{\boldsymbol{\phi}} \cos \phi) \frac{R^2 \sin^2 \theta \cos^2 \phi}{R^2 \sin^2 \theta \sin^2 \phi + R^2 \sin^2 \theta \cos^2 \phi} \\ &\quad + (\hat{\mathbf{R}} \cos \theta - \hat{\boldsymbol{\theta}} \sin \theta) 4 \\ &= \hat{\mathbf{R}} (\sin \theta \cos \phi \sin^2 \phi - \sin \theta \sin \phi \cos^2 \phi + 4 \cos \theta) \\ &\quad + \hat{\boldsymbol{\theta}} (\cos \theta \cos \phi \sin^2 \phi - \cos \theta \sin \phi \cos^2 \phi - 4 \sin \theta) \\ &\quad - \hat{\boldsymbol{\phi}} (\cos^3 \phi + \sin^3 \phi),\end{aligned}$$

$$\begin{aligned}P_4(1, -1, 2) &= P_4 \left[\sqrt{1+1+4}, \tan^{-1}(\sqrt{1+1}/2), \tan^{-1}(-1/1) \right] \\ &= P_4(\sqrt{6}, 35.26^\circ, -45^\circ),\end{aligned}$$

$$\begin{aligned}\mathbf{D}(P_4) &= \hat{\mathbf{R}} (\sin 35.26^\circ \cos 45^\circ \sin^2 45^\circ - \sin 35.26^\circ \sin(-45^\circ) \cos^2 45^\circ + 4 \cos 35.26^\circ) \\ &\quad + \hat{\boldsymbol{\theta}} (\cos 35.26^\circ \cos 45^\circ \sin^2 45^\circ - \cos 35.26^\circ \sin(-45^\circ) \cos^2 45^\circ - 4 \sin 35.26^\circ) \\ &\quad - \hat{\boldsymbol{\phi}} (\cos^3 45^\circ + \sin^3 45^\circ) \\ &= \hat{\mathbf{R}} 3.67 - \hat{\boldsymbol{\theta}} 1.73 - \hat{\boldsymbol{\phi}} 0.707.\end{aligned}$$
