

Problem 3.39 Follow a procedure similar to that leading to Eq. (3.82) to derive the expression given by Eq. (3.83) for ∇ in spherical coordinates.

Solution: From the chain rule and Table 3-2,

$$\begin{aligned}
 \nabla T &= \hat{\mathbf{x}} \frac{\partial T}{\partial x} + \hat{\mathbf{y}} \frac{\partial T}{\partial y} + \hat{\mathbf{z}} \frac{\partial T}{\partial z} \\
 &= \hat{\mathbf{x}} \left(\frac{\partial T}{\partial R} \frac{\partial R}{\partial x} + \frac{\partial T}{\partial \theta} \frac{\partial \theta}{\partial x} + \frac{\partial T}{\partial \phi} \frac{\partial \phi}{\partial x} \right) \\
 &\quad + \hat{\mathbf{y}} \left(\frac{\partial T}{\partial R} \frac{\partial R}{\partial y} + \frac{\partial T}{\partial \theta} \frac{\partial \theta}{\partial y} + \frac{\partial T}{\partial \phi} \frac{\partial \phi}{\partial y} \right) \\
 &\quad + \hat{\mathbf{z}} \left(\frac{\partial T}{\partial R} \frac{\partial R}{\partial z} + \frac{\partial T}{\partial \theta} \frac{\partial \theta}{\partial z} + \frac{\partial T}{\partial \phi} \frac{\partial \phi}{\partial z} \right) \\
 &= \hat{\mathbf{x}} \left(\frac{\partial T}{\partial R} \frac{\partial}{\partial x} \sqrt{x^2 + y^2 + z^2} + \frac{\partial T}{\partial \theta} \frac{\partial}{\partial x} \tan^{-1}(\sqrt{x^2 + y^2}/z) + \frac{\partial T}{\partial \phi} \frac{\partial}{\partial x} \tan^{-1}(y/x) \right) \\
 &\quad + \hat{\mathbf{y}} \left(\frac{\partial T}{\partial R} \frac{\partial}{\partial y} \sqrt{x^2 + y^2 + z^2} + \frac{\partial T}{\partial \theta} \frac{\partial}{\partial y} \tan^{-1}(\sqrt{x^2 + y^2}/z) + \frac{\partial T}{\partial \phi} \frac{\partial}{\partial y} \tan^{-1}(y/x) \right) \\
 &\quad + \hat{\mathbf{z}} \left(\frac{\partial T}{\partial R} \frac{\partial}{\partial z} \sqrt{x^2 + y^2 + z^2} + \frac{\partial T}{\partial \theta} \frac{\partial}{\partial z} \tan^{-1}(\sqrt{x^2 + y^2}/z) + \frac{\partial T}{\partial \phi} \frac{\partial}{\partial z} \tan^{-1}(y/x) \right) \\
 &= \hat{\mathbf{x}} \left(\frac{\partial T}{\partial R} \frac{x}{\sqrt{x^2 + y^2 + z^2}} + \frac{\partial T}{\partial \theta} \frac{z}{x^2 + y^2 + z^2} \frac{x}{\sqrt{x^2 + y^2}} + \frac{\partial T}{\partial \phi} \frac{-y}{x^2 + y^2} \right) \\
 &\quad + \hat{\mathbf{y}} \left(\frac{\partial T}{\partial R} \frac{y}{\sqrt{x^2 + y^2 + z^2}} + \frac{\partial T}{\partial \theta} \frac{z}{x^2 + y^2 + z^2} \frac{y}{\sqrt{x^2 + y^2}} + \frac{\partial T}{\partial \phi} \frac{x}{x^2 + y^2} \right) \\
 &\quad + \hat{\mathbf{z}} \left(\frac{\partial T}{\partial R} \frac{z}{\sqrt{x^2 + y^2 + z^2}} + \frac{\partial T}{\partial \theta} \frac{-1}{x^2 + y^2 + z^2} \sqrt{x^2 + y^2} + \frac{\partial T}{\partial \phi} 0 \right) \\
 &= \hat{\mathbf{x}} \left(\frac{\partial T}{\partial R} \frac{R \sin \theta \cos \phi}{R} + \frac{\partial T}{\partial \theta} \frac{R \cos \theta}{R^2} \frac{R \sin \theta \cos \phi}{R \sin \theta} + \frac{\partial T}{\partial \phi} \frac{-R \sin \theta \sin \phi}{R^2 \sin^2 \theta} \right) \\
 &\quad + \hat{\mathbf{y}} \left(\frac{\partial T}{\partial R} \frac{R \sin \theta \sin \phi}{R} + \frac{\partial T}{\partial \theta} \frac{R \cos \theta}{R^2} \frac{R \sin \theta \sin \phi}{R \sin \theta} + \frac{\partial T}{\partial \phi} \frac{R \sin \theta \cos \phi}{R^2 \sin^2 \theta} \right) \\
 &\quad + \hat{\mathbf{z}} \left(\frac{\partial T}{\partial R} \frac{R \cos \theta}{R} + \frac{\partial T}{\partial \theta} \frac{-R \sin \theta}{R^2} \right)
 \end{aligned}$$

$$\begin{aligned}
&= \hat{\mathbf{x}} \left(\frac{\partial T}{\partial R} \sin \theta \cos \phi + \frac{\partial T}{\partial \theta} \frac{\cos \theta \cos \phi}{R} + \frac{\partial T}{\partial \phi} \frac{-\sin \phi}{R \sin \theta} \right) \\
&\quad + \hat{\mathbf{y}} \left(\frac{\partial T}{\partial R} \sin \theta \sin \phi + \frac{\partial T}{\partial \theta} \frac{\cos \theta \sin \phi}{R} + \frac{\partial T}{\partial \phi} \frac{\cos \phi}{R \sin \theta} \right) \\
&\quad + \hat{\mathbf{z}} \left(\frac{\partial T}{\partial R} \cos \theta + \frac{\partial T}{\partial \theta} \frac{-\sin \theta}{R} \right) \\
&= (\hat{\mathbf{x}} \sin \theta \cos \phi + \hat{\mathbf{y}} \sin \theta \sin \phi + \hat{\mathbf{z}} \cos \theta) \frac{\partial T}{\partial R} \\
&\quad + (\hat{\mathbf{x}} \cos \theta \cos \phi + \hat{\mathbf{y}} \cos \theta \sin \phi - \hat{\mathbf{z}} \sin \theta) \frac{1}{R} \frac{\partial T}{\partial \theta} \\
&\quad + (-\hat{\mathbf{x}} \sin \phi + \hat{\mathbf{y}} \cos \phi) \frac{1}{R \sin \theta} \frac{\partial T}{\partial \phi} \\
&= \hat{\mathbf{R}} \frac{\partial T}{\partial R} + \hat{\boldsymbol{\theta}} \frac{1}{R} \frac{\partial T}{\partial \theta} + \hat{\boldsymbol{\phi}} \frac{1}{R \sin \theta} \frac{\partial T}{\partial \phi},
\end{aligned}$$

which is Eq. (3.83).
