Problem 3.44 Each of the following vector fields is displayed in Fig. P3.44 in the form of a vector representation. Determine $\nabla \cdot \mathbf{A}$ analytically and then compare the result with your expectations on the basis of the displayed pattern.

(a) $\mathbf{A} = -\hat{\mathbf{x}} \cos x \sin y + \hat{\mathbf{y}} \sin x \cos y$, for $-\pi \le x, y \le \pi$

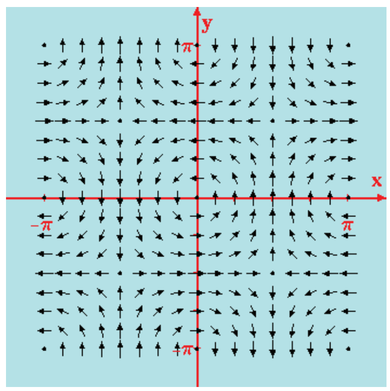

Figure P3.44(a)

Solution:

$$\mathbf{A} = -\hat{\mathbf{x}} \cos x \sin y + \hat{\mathbf{y}} \sin x \cos y$$

$$\begin{aligned}
\nabla \cdot \mathbf{A} &= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} \\
&= \frac{\partial}{\partial x}(-\cos x \sin y) + \frac{\partial}{\partial y}(-\sin x \cos y) \\
&= \sin x \sin y - \sin x \sin y = 0
\end{aligned}$$

Yes, $\mathbf{A}$ is divergenceless everywhere.

(b) $\mathbf{A} = -\hat{\mathbf{x}}\sin 2y + \hat{\mathbf{y}}\cos 2x$, for $-\pi \le x, y \le \pi$

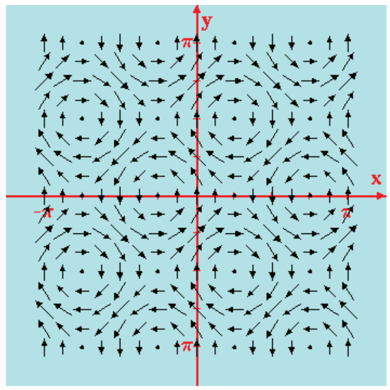

Figure P3.44(b)

Solution:

$$\mathbf{A} = -\hat{\mathbf{x}}\sin 2y + \hat{\mathbf{y}}\cos 2x$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y}$$

$$= \frac{\partial}{\partial x}(-\sin 2y) + \frac{\partial}{\partial y}(\cos 2x) = 0$$

Yes, **A** is divergenceless everywhere.

(c) $\mathbf{A} = -\hat{\mathbf{x}}xy + \hat{\mathbf{y}}y^2$, for $-10 \le x, y \le 10$

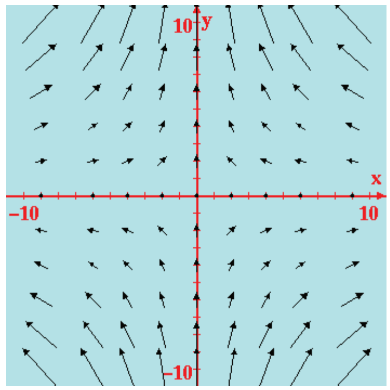

Figure P3.44(c)

Solution:

$$\mathbf{A} = -\hat{\mathbf{x}}xy + \hat{\mathbf{y}}y^2$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y}$$

$$= \frac{\partial}{\partial x}(-xy) + \frac{\partial}{\partial y}(y^2) = -y + 2y = y$$

NO, **A** is not divergenceless everywhere. It is divergenceless only at $y = 0$.

(d) $\mathbf{A} = -\hat{\mathbf{x}}\cos x + \hat{\mathbf{y}}\sin y$, for $-\pi \leq x, y \leq \pi$

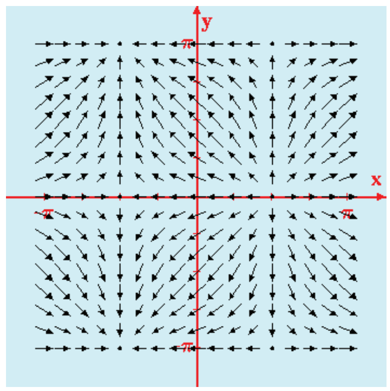

Figure P3.44(d)

Solution:

$$\mathbf{A} = -\hat{\mathbf{x}}\cos x + \hat{\mathbf{y}}\sin y$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y}$$

$$= \frac{\partial}{\partial x}(-\cos x) + \frac{\partial}{\partial y}(\sin y) = \sin x + \cos y$$

NO, **A** is not divergenceless everywhere.

(e) $\mathbf{A} = \hat{\mathbf{x}}x$, for $-10 \leq x \leq 10$

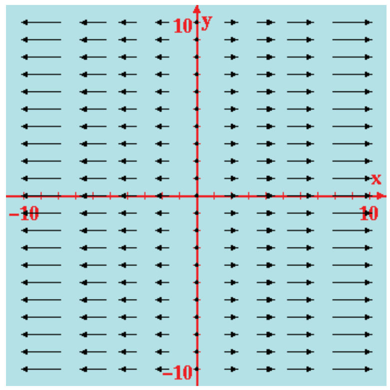

Figure P3.44(e)

Solution:

$$\mathbf{A} = \hat{\mathbf{x}}x$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$= 1$$

This indicates that the divergence of **A** is the same at all points in the defined space. In other words, every small volume is a source of flux (more flux leaving the volume than entering it), and the net generated flux is the same at all locations.

(f) $\mathbf{A} = \hat{\mathbf{x}}xy^2$, for $-10 \le x, y \le 10$

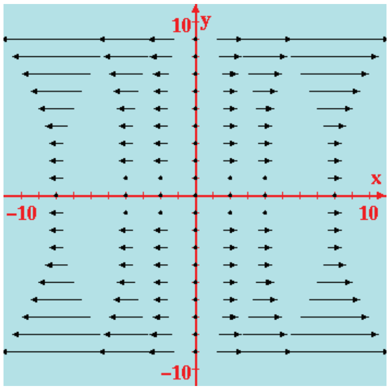

Figure P3.44(f)

Solution:

$$\mathbf{A} = \hat{\mathbf{x}}xy^2$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$= y^2$$

(g) $\mathbf{A} = \hat{\mathbf{x}}xy^2 + \hat{\mathbf{y}}x^2y$, for $-10 \le x, y \le 10$

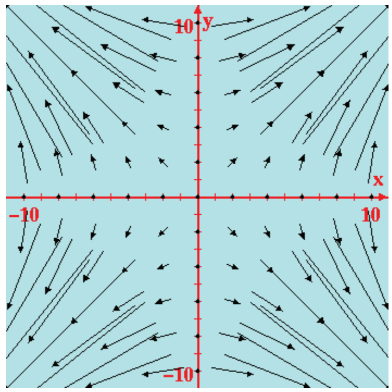

Figure P3.44(g)

Solution:

$$\mathbf{A} = \hat{\mathbf{x}}xy^2 + \hat{\mathbf{y}}x^2y$$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$= y^2 + x^2$$

(h) $\mathbf{A} = \hat{\mathbf{x}} \sin\left(\frac{\pi x}{10}\right) + \hat{\mathbf{y}} \sin\left(\frac{\pi y}{10}\right)$, for $-10 \leq x, y \leq 10$

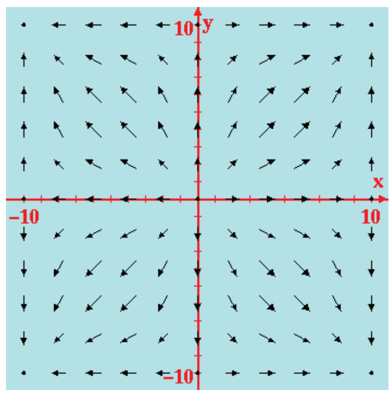

Figure P3.44(h)

Solution:

$$\mathbf{A} = \hat{\mathbf{x}} \sin(\pi x/10) + \hat{\mathbf{y}} \sin(\pi y/10)$$

$$\begin{aligned}
\nabla \cdot \mathbf{A} &= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \\
&= \frac{\pi}{10}[\cos(\pi x/10) + \cos(\pi y/10)]
\end{aligned}$$

(i) $\mathbf{A} = \hat{\mathbf{r}}\, r + \hat{\boldsymbol{\phi}}\, r\cos\phi$, for $\begin{cases} 0 \leq r \leq 10 \\ 0 \leq \phi \leq 2\pi. \end{cases}$

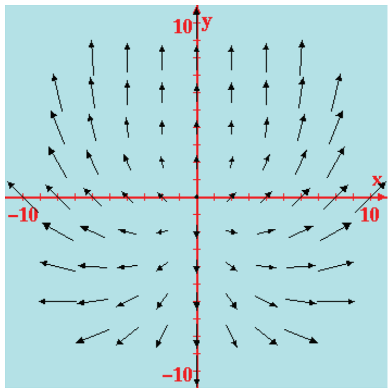

Figure P3.44(i)

Solution:

$$\begin{aligned}
\mathbf{A} &= \hat{\mathbf{r}}\, r + \hat{\boldsymbol{\phi}}\, r\cos\phi \\
\nabla\cdot\mathbf{A} &= \frac{1}{r}\,\frac{\partial}{\partial r}\,(rA_r) + \frac{1}{r}\,\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} \\
&= 2 - \sin\phi
\end{aligned}$$

(j) $\mathbf{A} = \hat{\mathbf{r}}\, r^2 + \hat{\boldsymbol{\phi}}\, r^2 \sin\phi$, for $\begin{cases} 0 \le r \le 10 \\ 0 \le \phi \le 2\pi. \end{cases}$

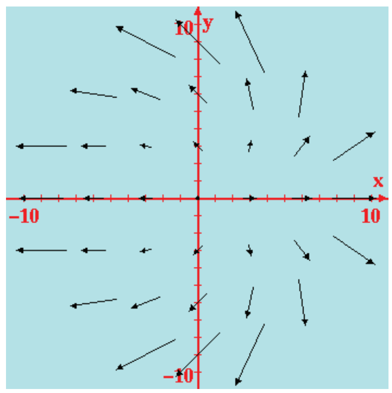

Figure P3.44(j)

Solution:

$$\mathbf{A} = \hat{\mathbf{r}}\, r^2 + \hat{\boldsymbol{\phi}}\, r^2 \sin\phi$$

$$\nabla \cdot \mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$= 3r + r\cos\phi$$