

**Problem 3.46** For the vector field  $\mathbf{E} = \hat{\mathbf{x}}xz - \hat{\mathbf{y}}yz^2 - \hat{\mathbf{z}}xy$ , verify the divergence theorem by computing:

- (a) the total outward flux flowing through the surface of a cube centered at the origin and with sides equal to 2 units each and parallel to the Cartesian axes, and
- (b) the integral of  $\nabla \cdot \mathbf{E}$  over the cube's volume.

**Solution:**

- (a) For a cube, the closed surface integral has 6 sides:

$$\begin{aligned}
 \oint \mathbf{E} \cdot d\mathbf{s} &= F_{\text{top}} + F_{\text{bottom}} + F_{\text{right}} + F_{\text{left}} + F_{\text{front}} + F_{\text{back}}, \\
 F_{\text{top}} &= \int_{x=-1}^1 \int_{y=-1}^1 (\hat{\mathbf{x}}xz - \hat{\mathbf{y}}yz^2 - \hat{\mathbf{z}}xy) \Big|_{z=1} \cdot (\hat{\mathbf{z}} dy dx) \\
 &= - \int_{x=-1}^1 \int_{y=-1}^1 xy dy dx = \left( \left( \frac{x^2 y^2}{4} \right) \Big|_{y=-1}^1 \right) \Big|_{x=-1}^1 = 0, \\
 F_{\text{bottom}} &= \int_{x=-1}^1 \int_{y=-1}^1 (\hat{\mathbf{x}}xz - \hat{\mathbf{y}}yz^2 - \hat{\mathbf{z}}xy) \Big|_{z=-1} \cdot (-\hat{\mathbf{z}} dy dx) \\
 &= \int_{x=-1}^1 \int_{y=-1}^1 xy dy dx = \left( \left( \frac{x^2 y^2}{4} \right) \Big|_{y=-1}^1 \right) \Big|_{x=-1}^1 = 0, \\
 F_{\text{right}} &= \int_{x=-1}^1 \int_{z=-1}^1 (\hat{\mathbf{x}}xz - \hat{\mathbf{y}}yz^2 - \hat{\mathbf{z}}xy) \Big|_{y=1} \cdot (\hat{\mathbf{y}} dz dx) \\
 &= - \int_{x=-1}^1 \int_{z=-1}^1 z^2 dz dx = - \left( \left( \frac{xz^3}{3} \right) \Big|_{z=-1}^1 \right) \Big|_{x=-1}^1 = \frac{-4}{3}, \\
 F_{\text{left}} &= \int_{x=-1}^1 \int_{z=-1}^1 (\hat{\mathbf{x}}xz - \hat{\mathbf{y}}yz^2 - \hat{\mathbf{z}}xy) \Big|_{y=-1} \cdot (-\hat{\mathbf{y}} dz dx) \\
 &= - \int_{x=-1}^1 \int_{z=-1}^1 z^2 dz dx = - \left( \left( \frac{xz^3}{3} \right) \Big|_{z=-1}^1 \right) \Big|_{x=-1}^1 = \frac{-4}{3}, \\
 F_{\text{front}} &= \int_{y=-1}^1 \int_{z=-1}^1 (\hat{\mathbf{x}}xz - \hat{\mathbf{y}}yz^2 - \hat{\mathbf{z}}xy) \Big|_{x=1} \cdot (\hat{\mathbf{x}} dz dy) \\
 &= \int_{y=-1}^1 \int_{z=-1}^1 z dz dy = \left( \left( \frac{yz^2}{2} \right) \Big|_{z=-1}^1 \right) \Big|_{y=-1}^1 = 0,
 \end{aligned}$$

$$\begin{aligned}
F_{\text{back}} &= \int_{y=-1}^1 \int_{z=-1}^1 (\hat{\mathbf{x}}xz - \hat{\mathbf{y}}yz^2 - \hat{\mathbf{z}}xy) \Big|_{x=-1} \cdot (-\hat{\mathbf{x}} dz dy) \\
&= \int_{y=-1}^1 \int_{z=-1}^1 z dz dy = \left( \left( \frac{yz^2}{2} \right) \Big|_{z=-1}^1 \right) \Big|_{y=-1}^1 = 0, \\
\oint \mathbf{E} \cdot d\mathbf{s} &= 0 + 0 + \frac{-4}{3} + \frac{-4}{3} + 0 + 0 = \frac{-8}{3}.
\end{aligned}$$

(b)

$$\begin{aligned}
\iiint \nabla \cdot \mathbf{E} dv &= \int_{x=-1}^1 \int_{y=-1}^1 \int_{z=-1}^1 \nabla \cdot (\hat{\mathbf{x}}xz - \hat{\mathbf{y}}yz^2 - \hat{\mathbf{z}}xy) dz dy dx \\
&= \int_{x=-1}^1 \int_{y=-1}^1 \int_{z=-1}^1 (z - z^2) dz dy dx \\
&= \left( \left( \left( xy \left( \frac{z^2}{2} - \frac{z^3}{3} \right) \right) \Big|_{z=-1}^1 \right) \Big|_{y=-1}^1 \right) \Big|_{x=-1}^1 = \frac{-8}{3}.
\end{aligned}$$


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