

Problem 3.47 For the vector field $\mathbf{E} = \hat{\mathbf{r}}10e^{-r} - \hat{\mathbf{z}}3z$, verify the divergence theorem for the cylindrical region enclosed by $r = 2$, $z = 0$, and $z = 4$.

Solution:

$$\begin{aligned}
 \oint \mathbf{E} \cdot d\mathbf{s} &= \int_{r=0}^2 \int_{\phi=0}^{2\pi} ((\hat{\mathbf{r}}10e^{-r} - \hat{\mathbf{z}}3z) \cdot (-\hat{\mathbf{z}}r dr d\phi)) \Big|_{z=0} \\
 &\quad + \int_{\phi=0}^{2\pi} \int_{z=0}^4 ((\hat{\mathbf{r}}10e^{-r} - \hat{\mathbf{z}}3z) \cdot (\hat{\mathbf{r}}r d\phi dz)) \Big|_{r=2} \\
 &\quad + \int_{r=0}^2 \int_{\phi=0}^{2\pi} ((\hat{\mathbf{r}}10e^{-r} - \hat{\mathbf{z}}3z) \cdot (\hat{\mathbf{z}}r dr d\phi)) \Big|_{z=4} \\
 &= 0 + \int_{\phi=0}^{2\pi} \int_{z=0}^4 10e^{-2} 2 d\phi dz + \int_{r=0}^2 \int_{\phi=0}^{2\pi} -12r dr d\phi \\
 &= 160\pi e^{-2} - 48\pi \approx -82.77, \\
 \iiint \nabla \cdot \mathbf{E} d\mathcal{V} &= \int_{z=0}^4 \int_{r=0}^2 \int_{\phi=0}^{2\pi} \left(\frac{10e^{-r}(1-r)}{r} - 3 \right) r d\phi dr dz \\
 &= 8\pi \int_{r=0}^2 (10e^{-r}(1-r) - 3r) dr \\
 &= 8\pi \left(-10e^{-r} + 10e^{-r}(1+r) - \frac{3r^2}{2} \right) \Big|_{r=0}^2 \\
 &= 160\pi e^{-2} - 48\pi \approx -82.77.
 \end{aligned}$$
