

**Problem 3.48** A vector field  $\mathbf{D} = \hat{\mathbf{r}}r^3$  exists in the region between two concentric cylindrical surfaces defined by  $r = 1$  and  $r = 2$ , with both cylinders extending between  $z = 0$  and  $z = 5$ . Verify the divergence theorem by evaluating:

- (a)  $\oint_S \mathbf{D} \cdot d\mathbf{s}$ ,  
 (b)  $\int_{\mathcal{V}} \nabla \cdot \mathbf{D} d\mathcal{V}$ .

**Solution:**

(a)

$$\begin{aligned} \iint \mathbf{D} \cdot d\mathbf{s} &= F_{\text{inner}} + F_{\text{outer}} + F_{\text{bottom}} + F_{\text{top}}, \\ F_{\text{inner}} &= \int_{\phi=0}^{2\pi} \int_{z=0}^5 ((\hat{\mathbf{r}}r^3) \cdot (-\hat{\mathbf{r}}r dz d\phi)) \Big|_{r=1} \\ &= \int_{\phi=0}^{2\pi} \int_{z=0}^5 (-r^4 dz d\phi) \Big|_{r=1} = -10\pi, \\ F_{\text{outer}} &= \int_{\phi=0}^{2\pi} \int_{z=0}^5 ((\hat{\mathbf{r}}r^3) \cdot (\hat{\mathbf{r}}r dz d\phi)) \Big|_{r=2} \\ &= \int_{\phi=0}^{2\pi} \int_{z=0}^5 (r^4 dz d\phi) \Big|_{r=2} = 160\pi, \\ F_{\text{bottom}} &= \int_{r=1}^2 \int_{\phi=0}^{2\pi} ((\hat{\mathbf{r}}r^3) \cdot (-\hat{\mathbf{z}}r d\phi dr)) \Big|_{z=0} = 0, \\ F_{\text{top}} &= \int_{r=1}^2 \int_{\phi=0}^{2\pi} ((\hat{\mathbf{r}}r^3) \cdot (\hat{\mathbf{z}}r d\phi dr)) \Big|_{z=5} = 0. \end{aligned}$$

Therefore,  $\iint \mathbf{D} \cdot d\mathbf{s} = 150\pi$ .

(b) From the back cover,  $\nabla \cdot \mathbf{D} = (1/r)(\partial/\partial r)(rr^3) = 4r^2$ . Therefore,

$$\iiint \nabla \cdot \mathbf{D} d\mathcal{V} = \int_{z=0}^5 \int_{\phi=0}^{2\pi} \int_{r=1}^2 4r^2 r dr d\phi dz = \left( (r^4) \Big|_{r=1}^2 \right) \Big|_{\phi=0}^{2\pi} \Big|_{z=0}^5 = 150\pi.$$


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