

Problem 3.50 For the vector field $\mathbf{E} = \hat{\mathbf{x}}xy - \hat{\mathbf{y}}(x^2 + 2y^2)$, calculate

- (a) $\oint_C \mathbf{E} \cdot d\mathbf{l}$ around the triangular contour shown in Fig. P3.50(a), and
 (b) $\int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{s}$ over the area of the triangle.

Solution: In addition to the independent condition that $z = 0$, the three lines of the triangle are represented by the equations $y = 0$, $x = 1$, and $y = x$, respectively.

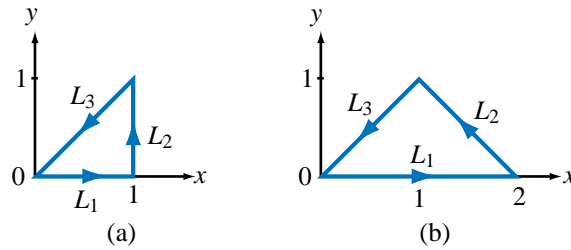


Figure P3.50: Contours for (a) Problem 3.50 and (b) Problem 3.51.

(a)

$$\begin{aligned}
 \oint \mathbf{E} \cdot d\mathbf{l} &= L_1 + L_2 + L_3, \\
 L_1 &= \int (\hat{\mathbf{x}}xy - \hat{\mathbf{y}}(x^2 + 2y^2)) \cdot (\hat{\mathbf{x}}dx + \hat{\mathbf{y}}dy + \hat{\mathbf{z}}dz) \\
 &= \int_{x=0}^1 (xy)|_{y=0, z=0} dx - \int_{y=0}^0 (x^2 + 2y^2)|_{z=0} dy + \int_{z=0}^0 (0)|_{y=0} dz = 0, \\
 L_2 &= \int (\hat{\mathbf{x}}xy - \hat{\mathbf{y}}(x^2 + 2y^2)) \cdot (\hat{\mathbf{x}}dx + \hat{\mathbf{y}}dy + \hat{\mathbf{z}}dz) \\
 &= \int_{x=1}^1 (xy)|_{z=0} dx - \int_{y=0}^1 (x^2 + 2y^2)|_{x=1, z=0} dy + \int_{z=0}^0 (0)|_{x=1} dz \\
 &= 0 - \left(y + \frac{2y^3}{3} \right) \Big|_{y=0}^1 + 0 = -\frac{5}{3}, \\
 L_3 &= \int (\hat{\mathbf{x}}xy - \hat{\mathbf{y}}(x^2 + 2y^2)) \cdot (\hat{\mathbf{x}}dx + \hat{\mathbf{y}}dy + \hat{\mathbf{z}}dz) \\
 &= \int_{x=1}^0 (xy)|_{y=x, z=0} dx - \int_{y=1}^0 (x^2 + 2y^2)|_{x=y, z=0} dy + \int_{z=0}^0 (0)|_{y=x} dz \\
 &= \left(\frac{x^3}{3} \right) \Big|_{x=1}^0 - (y^3) \Big|_{y=1}^0 + 0 = \frac{2}{3}.
 \end{aligned}$$

Therefore,

$$\oint \mathbf{E} \cdot d\mathbf{l} = 0 - \frac{5}{3} + \frac{2}{3} = -1.$$

(b) From Eq. (3.105), $\nabla \times \mathbf{E} = -\hat{\mathbf{z}}3x$, so that

$$\begin{aligned} \iint \nabla \times \mathbf{E} \cdot d\mathbf{s} &= \int_{x=0}^1 \int_{y=0}^x ((-\hat{\mathbf{z}}3x) \cdot (\hat{\mathbf{z}} dy dx))|_{z=0} \\ &= - \int_{x=0}^1 \int_{y=0}^x 3x dy dx = - \int_{x=0}^1 3x(x-0) dx = -(x^3)|_0^1 = -1. \end{aligned}$$
