

Problem 3.51 Repeat Problem 3.50 for the contour shown in Fig. P3.50(b).

Solution: In addition to the independent condition that $z = 0$, the three lines of the triangle are represented by the equations $y = 0$, $y = 2 - x$, and $y = x$, respectively.

(a)

$$\oint \mathbf{E} \cdot d\mathbf{l} = L_1 + L_2 + L_3,$$

$$\begin{aligned} L_1 &= \int (\hat{\mathbf{x}}xy - \hat{\mathbf{y}}(x^2 + 2y^2)) \cdot (\hat{\mathbf{x}}dx + \hat{\mathbf{y}}dy + \hat{\mathbf{z}}dz) \\ &= \int_{x=0}^2 (xy)|_{y=0, z=0} dx - \int_{y=0}^0 (x^2 + 2y^2)|_{z=0} dy + \int_{z=0}^0 (0)|_{y=0} dz = 0, \\ L_2 &= \int (\hat{\mathbf{x}}xy - \hat{\mathbf{y}}(x^2 + 2y^2)) \cdot (\hat{\mathbf{x}}dx + \hat{\mathbf{y}}dy + \hat{\mathbf{z}}dz) \\ &= \int_{x=2}^1 (xy)|_{z=0, y=2-x} dx - \int_{y=0}^1 (x^2 + 2y^2)|_{x=2-y, z=0} dy + \int_{z=0}^0 (0)|_{y=2-x} dz \\ &= \left(x^2 - \frac{x^3}{3}\right)\Big|_{x=2}^1 - (4y - 2y^2 + y^3)\Big|_{y=0}^1 + 0 = \frac{-11}{3}, \\ L_3 &= \int (\hat{\mathbf{x}}xy - \hat{\mathbf{y}}(x^2 + 2y^2)) \cdot (\hat{\mathbf{x}}dx + \hat{\mathbf{y}}dy + \hat{\mathbf{z}}dz) \\ &= \int_{x=1}^0 (xy)|_{y=x, z=0} dx - \int_{y=1}^0 (x^2 + 2y^2)|_{x=y, z=0} dy + \int_{z=0}^0 (0)|_{y=x} dz \\ &= \left(\frac{x^3}{3}\right)\Big|_{x=1}^0 - (y^3)\Big|_{y=1}^0 + 0 = \frac{2}{3}. \end{aligned}$$

Therefore,

$$\oint \mathbf{E} \cdot d\mathbf{l} = 0 - \frac{11}{3} + \frac{2}{3} = -3.$$

(b) From Eq. (3.105), $\nabla \times \mathbf{E} = -\hat{\mathbf{z}}3x$, so that

$$\begin{aligned} \iint \nabla \times \mathbf{E} \cdot d\mathbf{s} &= \int_{x=0}^1 \int_{y=0}^x ((-\hat{\mathbf{z}}3x) \cdot (\hat{\mathbf{z}}dy dx))|_{z=0} \\ &\quad + \int_{x=1}^2 \int_{y=0}^{2-x} ((-\hat{\mathbf{z}}3x) \cdot (\hat{\mathbf{z}}dy dx))|_{z=0} \\ &= - \int_{x=0}^1 \int_{y=0}^x 3x dy dx - \int_{x=1}^2 \int_{y=0}^{2-x} 3x dy dx \\ &= - \int_{x=0}^1 3x(x-0) dx - \int_{x=1}^2 3x((2-x)-0) dx \\ &= -(x^3)\Big|_0^1 - (3x^2 - x^3)\Big|_1^2 = -3. \end{aligned}$$
