

Problem 3.52 Verify Stokes's theorem for the vector field $\mathbf{B} = (\hat{\mathbf{r}}r \cos \phi + \hat{\boldsymbol{\phi}} \sin \phi)$ by evaluating:

- (a) $\oint_C \mathbf{B} \cdot d\mathbf{l}$ over the semicircular contour shown in Fig. P3.52(a), and
(b) $\int_S (\nabla \times \mathbf{B}) \cdot d\mathbf{s}$ over the surface of the semicircle.

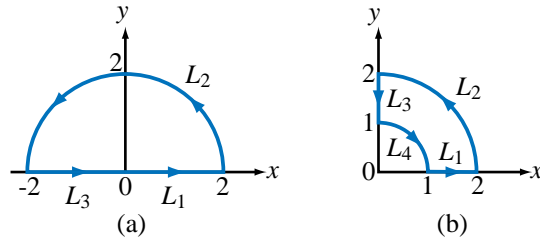


Figure P3.52: Contour paths for (a) Problem 3.52 and (b) Problem 3.53.

Solution:

(a)

$$\begin{aligned}
 \oint_C \mathbf{B} \cdot d\mathbf{l} &= \int_{L_1} \mathbf{B} \cdot d\mathbf{l} + \int_{L_2} \mathbf{B} \cdot d\mathbf{l} + \int_{L_3} \mathbf{B} \cdot d\mathbf{l}, \\
 \mathbf{B} \cdot d\mathbf{l} &= (\hat{\mathbf{r}}r \cos \phi + \hat{\boldsymbol{\phi}} \sin \phi) \cdot (\hat{\mathbf{r}} dr + \hat{\boldsymbol{\phi}} r d\phi + \hat{\mathbf{z}} dz) = r \cos \phi dr + r \sin \phi d\phi, \\
 \int_{L_1} \mathbf{B} \cdot d\mathbf{l} &= \left(\int_{r=0}^2 r \cos \phi dr \right) \Big|_{\phi=0, z=0} + \left(\int_{\phi=0}^0 r \sin \phi d\phi \right) \Big|_{z=0} \\
 &= \left(\frac{1}{2} r^2 \right) \Big|_{r=0}^2 + 0 = 2, \\
 \int_{L_2} \mathbf{B} \cdot d\mathbf{l} &= \left(\int_{r=2}^2 r \cos \phi dr \right) \Big|_{z=0} + \left(\int_{\phi=0}^{\pi} r \sin \phi d\phi \right) \Big|_{r=2, z=0} \\
 &= 0 + (-2 \cos \phi) \Big|_{\phi=0}^{\pi} = 4, \\
 \int_{L_3} \mathbf{B} \cdot d\mathbf{l} &= \left(\int_{r=2}^0 r \cos \phi dr \right) \Big|_{\phi=\pi, z=0} + \left(\int_{\phi=\pi}^{\pi} r \sin \phi d\phi \right) \Big|_{z=0} \\
 &= \left(-\frac{1}{2} r^2 \right) \Big|_{r=2}^0 + 0 = 2, \\
 \oint_C \mathbf{B} \cdot d\mathbf{l} &= 2 + 4 + 2 = 8.
 \end{aligned}$$

(b)

$$\begin{aligned}\nabla \times \mathbf{B} &= \nabla \times (\hat{\mathbf{r}} r \cos \phi + \hat{\boldsymbol{\phi}} \sin \phi) \\&= \hat{\mathbf{r}} \left(\frac{1}{r} \frac{\partial}{\partial \phi} 0 - \frac{\partial}{\partial z} (\sin \phi) \right) + \hat{\boldsymbol{\phi}} \left(\frac{\partial}{\partial z} (r \cos \phi) - \frac{\partial}{\partial r} 0 \right) \\&\quad + \hat{\mathbf{z}} \frac{1}{r} \left(\frac{\partial}{\partial r} (r (\sin \phi)) - \frac{\partial}{\partial \phi} (r \cos \phi) \right) \\&= \hat{\mathbf{r}} 0 + \hat{\boldsymbol{\phi}} 0 + \hat{\mathbf{z}} \frac{1}{r} (\sin \phi + (r \sin \phi)) = \hat{\mathbf{z}} \sin \phi \left(1 + \frac{1}{r} \right), \\ \iint \nabla \times \mathbf{B} \cdot d\mathbf{s} &= \int_{\phi=0}^{\pi} \int_{r=0}^2 \left(\hat{\mathbf{z}} \sin \phi \left(1 + \frac{1}{r} \right) \right) \cdot (\hat{\mathbf{z}} r dr d\phi) \\&= \int_{\phi=0}^{\pi} \int_{r=0}^2 \sin \phi (r+1) dr d\phi = \left((-\cos \phi (\tfrac{1}{2} r^2 + r)) \Big|_{r=0}^2 \right) \Big|_{\phi=0}^{\pi} = 8.\end{aligned}$$
