

Problem 3.54 Verify Stokes's Theorem for the vector field $\mathbf{A} = \hat{\mathbf{R}} \cos \theta + \hat{\boldsymbol{\phi}} \sin \theta$ by evaluating it on the hemisphere of unit radius.

Solution:

$$\mathbf{A} = \hat{\mathbf{R}} \cos \theta + \hat{\boldsymbol{\phi}} \sin \theta = \hat{\mathbf{R}} A_R + \hat{\boldsymbol{\theta}} A_\theta + \hat{\boldsymbol{\phi}} A_\phi.$$

Hence, $A_R = \cos \theta$, $A_\theta = 0$, $A_\phi = \sin \theta$.

$$\begin{aligned} \nabla \times \mathbf{A} &= \hat{\mathbf{R}} \frac{1}{R \sin \theta} \left(\frac{\partial}{\partial \theta} (A_\phi \sin \theta) \right) - \hat{\boldsymbol{\theta}} \frac{1}{R} \frac{\partial}{\partial R} (R A_\phi) - \hat{\boldsymbol{\phi}} \frac{1}{R} \frac{\partial A_R}{\partial \theta} \\ &= \hat{\mathbf{R}} \frac{1}{R \sin \theta} \frac{\partial}{\partial \theta} (\sin^2 \theta) - \hat{\boldsymbol{\theta}} \frac{1}{R} \frac{\partial}{\partial R} (R \sin \theta) - \hat{\boldsymbol{\phi}} \frac{1}{R} \frac{\partial}{\partial \theta} (\cos \theta) \\ &= \hat{\mathbf{R}} \frac{2 \cos \theta}{R} - \hat{\boldsymbol{\theta}} \frac{\sin \theta}{R} + \hat{\boldsymbol{\phi}} \frac{\sin \theta}{R}. \end{aligned}$$

For the hemispherical surface, $d\mathbf{s} = \hat{\mathbf{R}} R^2 \sin \theta d\theta d\phi$.

$$\begin{aligned} &\int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/2} (\nabla \times \mathbf{A}) \cdot d\mathbf{s} \\ &= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/2} \left(\frac{\hat{\mathbf{R}} 2 \cos \theta}{R} - \hat{\boldsymbol{\theta}} \frac{\sin \theta}{R} + \hat{\boldsymbol{\phi}} \frac{\sin \theta}{R} \right) \cdot \hat{\mathbf{R}} R^2 \sin \theta d\theta d\phi \Big|_{R=1} \\ &= 4\pi R \frac{\sin^2 \theta}{2} \Big|_0^{\pi/2} \Big|_{R=1} = 2\pi. \end{aligned}$$

The contour C is the circle in the x - y plane bounding the hemispherical surface.

$$\oint_C \mathbf{A} \cdot d\mathbf{l} = \int_{\phi=0}^{2\pi} (\hat{\mathbf{R}} \cos \theta + \hat{\boldsymbol{\phi}} \sin \theta) \cdot \hat{\boldsymbol{\phi}} R d\phi \Big|_{\theta=\pi/2}^{R=1} = R \sin \theta \int_0^{2\pi} d\phi \Big|_{\theta=\pi/2}^{R=1} = 2\pi.$$
