

Problem 3.55 Verify Stokes's theorem for the vector field $\mathbf{B} = (\hat{\mathbf{r}} \cos \phi + \hat{\boldsymbol{\phi}} \sin \phi)$ by evaluating:

- (a) $\oint_C \mathbf{B} \cdot d\boldsymbol{\ell}$ over the path comprising a quarter section of a circle, as shown in Fig. P3.55, and
 (b) $\int_S (\nabla \times \mathbf{B}) \cdot d\mathbf{s}$ over the surface of the quarter section.

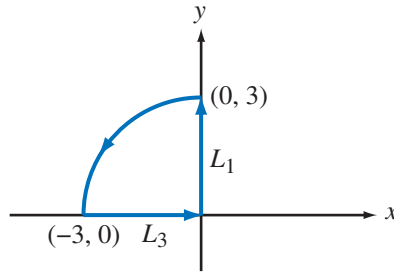


Figure P3.55: Problem 3.55.

Solution:

(a)

$$\oint_C \mathbf{B} \cdot d\boldsymbol{\ell} = \int_{L_1} \mathbf{B} \cdot d\boldsymbol{\ell} + \int_{L_2} \mathbf{B} \cdot d\boldsymbol{\ell} + \int_{L_3} \mathbf{B} \cdot d\boldsymbol{\ell}$$

Given the shape of the path, it is best to use cylindrical coordinates. \mathbf{B} is already expressed in cylindrical coordinates, and we need to choose $d\boldsymbol{\ell}$ in cylindrical coordinates:

$$d\boldsymbol{\ell} = \hat{\mathbf{r}} dr + \hat{\boldsymbol{\phi}} r d\phi + \hat{\mathbf{z}} dz.$$

Along path L_1 , $d\phi = 0$ and $dz = 0$. Hence, $d\boldsymbol{\ell} = \hat{\mathbf{r}} dr$ and

$$\begin{aligned} \int_{L_1} \mathbf{B} \cdot d\boldsymbol{\ell} &= \int_{r=0}^{r=3} (\hat{\mathbf{r}} \cos \phi + \hat{\boldsymbol{\phi}} \sin \phi) \cdot \hat{\mathbf{r}} dr \Big|_{\phi=90^\circ} \\ &= \int_{r=0}^3 \cos \phi dr \Big|_{\phi=90^\circ} = r \cos \phi \Big|_{r=0}^3 \Big|_{\phi=90^\circ} = 0. \end{aligned}$$

Along L_2 , $dr = dz = 0$. Hence, $d\boldsymbol{\ell} = \hat{\boldsymbol{\phi}} r d\phi$ and

$$\begin{aligned} \int_{L_2} \mathbf{B} \cdot d\boldsymbol{\ell} &= \int_{\phi=90^\circ}^{\phi=180^\circ} (\hat{\mathbf{r}} \cos \phi + \hat{\boldsymbol{\phi}} \sin \phi) \cdot \hat{\boldsymbol{\phi}} r d\phi \Big|_{r=3} \\ &= -3 \cos \phi \Big|_{90^\circ}^{180^\circ} = 3. \end{aligned}$$

Along L_3 , $dz = 0$ and $d\phi = 0$. Hence, $d\ell = \hat{\mathbf{r}} dr$ and

$$\begin{aligned}\int_{L_3} \mathbf{B} \cdot d\ell &= \int_{r=3}^0 (\hat{\mathbf{r}} \cos \phi + \hat{\phi} \sin \phi) \cdot \hat{\mathbf{r}} dr \Big|_{\phi=180^\circ} \\ &= \int_{r=3}^0 \cos \phi dr \Big|_{\phi=180^\circ} = -r \Big|_3^0 = 3.\end{aligned}$$

Hence,

$$\oint_C \mathbf{B} \cdot d\ell = 0 + 3 + 3 = 6.$$

(b)

$$\begin{aligned}\nabla \times \mathbf{B} &= \hat{\mathbf{z}} \frac{1}{r} \left(\frac{\partial}{\partial r} \left(r B_\phi - \frac{\partial B_r}{\partial \phi} \right) \right) \\ &= \hat{\mathbf{z}} \frac{1}{r} \left(\frac{\partial}{\partial r} (r \sin \phi) - \frac{\partial}{\partial \phi} (\cos \phi) \right) \\ &= \hat{\mathbf{z}} \frac{1}{r} (\sin \phi + \sin \phi) = \hat{\mathbf{z}} \frac{2}{r} \sin \phi. \\ \int_S (\nabla \times \mathbf{B}) \cdot d\mathbf{s} &= \int_{r=0}^3 \int_{\phi=90^\circ}^{180^\circ} \left(\hat{\mathbf{z}} \frac{2}{r} \sin \phi \right) \cdot \hat{\mathbf{z}} r dr d\phi \\ &= -2r \Big|_{r=0}^3 \cos \phi \Big|_{\phi=90^\circ}^{180^\circ} = 6.\end{aligned}$$

Hence, Stokes's theorem is verified.
