

Problem 3.56 Determine if each of the following vector fields is solenoidal, conservative, or both:

- (a) $\mathbf{A} = \hat{\mathbf{x}}x^2 - \hat{\mathbf{y}}y2xy,$
- (b) $\mathbf{B} = \hat{\mathbf{x}}x^2 - \hat{\mathbf{y}}y^2 + \hat{\mathbf{z}}2z,$
- (c) $\mathbf{C} = \hat{\mathbf{r}}(\sin \phi)/r^2 + \hat{\boldsymbol{\phi}}(\cos \phi)/r^2,$
- (d) $\mathbf{D} = \hat{\mathbf{R}}/R,$
- (e) $\mathbf{E} = \hat{\mathbf{r}}\left(3 - \frac{r}{1+r}\right) + \hat{\mathbf{z}}z,$
- (f) $\mathbf{F} = (\hat{\mathbf{x}}y + \hat{\mathbf{y}}x)/(x^2 + y^2),$
- (g) $\mathbf{G} = \hat{\mathbf{x}}(x^2 + z^2) - \hat{\mathbf{y}}(y^2 + x^2) - \hat{\mathbf{z}}(y^2 + z^2),$
- (h) $\mathbf{H} = \hat{\mathbf{R}}(Re^{-R}).$

Solution:

(a)

$$\nabla \cdot \mathbf{A} = \nabla \cdot (\hat{\mathbf{x}}x^2 - \hat{\mathbf{y}}2xy) = \frac{\partial}{\partial x}x^2 - \frac{\partial}{\partial y}2xy = 2x - 2x = 0,$$

$$\begin{aligned}\nabla \times \mathbf{A} &= \nabla \times (\hat{\mathbf{x}}x^2 - \hat{\mathbf{y}}2xy) \\ &= \hat{\mathbf{x}} \left(\frac{\partial}{\partial y}0 - \frac{\partial}{\partial z}(-2xy) \right) + \hat{\mathbf{y}} \left(\frac{\partial}{\partial z}(x^2) - \frac{\partial}{\partial x}0 \right) + \hat{\mathbf{z}} \left(\frac{\partial}{\partial x}(-2xy) - \frac{\partial}{\partial y}(x^2) \right) \\ &= \hat{\mathbf{x}}0 + \hat{\mathbf{y}}0 - \hat{\mathbf{z}}(2y) \neq 0.\end{aligned}$$

The field \mathbf{A} is solenoidal but not conservative.

(b)

$$\nabla \cdot \mathbf{B} = \nabla \cdot (\hat{\mathbf{x}}x^2 - \hat{\mathbf{y}}y^2 + \hat{\mathbf{z}}2z) = \frac{\partial}{\partial x}x^2 - \frac{\partial}{\partial y}y^2 + \frac{\partial}{\partial z}2z = 2x - 2y + 2 \neq 0,$$

$$\begin{aligned}\nabla \times \mathbf{B} &= \nabla \times (\hat{\mathbf{x}}x^2 - \hat{\mathbf{y}}y^2 + \hat{\mathbf{z}}2z) \\ &= \hat{\mathbf{x}} \left(\frac{\partial}{\partial y}(2z) - \frac{\partial}{\partial z}(-y^2) \right) + \hat{\mathbf{y}} \left(\frac{\partial}{\partial z}(x^2) - \frac{\partial}{\partial x}(2z) \right) \\ &\quad + \hat{\mathbf{z}} \left(\frac{\partial}{\partial x}(-y^2) - \frac{\partial}{\partial y}(x^2) \right) \\ &= \hat{\mathbf{x}}0 + \hat{\mathbf{y}}0 + \hat{\mathbf{z}}0.\end{aligned}$$

The field \mathbf{B} is conservative but not solenoidal.

(c)

$$\begin{aligned}\nabla \cdot \mathbf{C} &= \nabla \cdot \left(\hat{\mathbf{r}} \frac{\sin \phi}{r^2} + \hat{\boldsymbol{\phi}} \frac{\cos \phi}{r^2} \right) \\ &= \frac{1}{r} \frac{\partial}{\partial r} \left(r \left(\frac{\sin \phi}{r^2} \right) \right) + \frac{1}{r} \frac{\partial}{\partial \phi} \left(\frac{\cos \phi}{r^2} \right) + \frac{\partial}{\partial z}0 \\ &= \frac{-\sin \phi}{r^3} + \frac{-\sin \phi}{r^3} + 0 = \frac{-2\sin \phi}{r^3},\end{aligned}$$

$$\begin{aligned}
\nabla \times \mathbf{C} &= \nabla \times \left(\hat{\mathbf{r}} \frac{\sin \phi}{r^2} + \hat{\boldsymbol{\phi}} \frac{\cos \phi}{r^2} \right) \\
&= \hat{\mathbf{r}} \left(\frac{1}{r} \frac{\partial}{\partial \phi} 0 - \frac{\partial}{\partial z} \left(\frac{\cos \phi}{r^2} \right) \right) + \hat{\boldsymbol{\phi}} \left(\frac{\partial}{\partial z} \left(\frac{\sin \phi}{r^2} \right) - \frac{\partial}{\partial r} 0 \right) \\
&\quad + \hat{\mathbf{z}} \frac{1}{r} \left(\frac{\partial}{\partial r} \left(r \left(\frac{\cos \phi}{r^2} \right) \right) - \frac{\partial}{\partial \phi} \left(\frac{\sin \phi}{r^2} \right) \right) \\
&= \hat{\mathbf{r}} 0 + \hat{\boldsymbol{\phi}} 0 + \hat{\mathbf{z}} \frac{1}{r} \left(- \left(\frac{\cos \phi}{r^2} \right) - \left(\frac{\cos \phi}{r^2} \right) \right) = \hat{\mathbf{z}} \frac{-2 \cos \phi}{r^3}.
\end{aligned}$$

The field \mathbf{C} is neither solenoidal nor conservative.

(d)

$$\begin{aligned}
\nabla \cdot \mathbf{D} &= \nabla \cdot \left(\frac{\hat{\mathbf{R}}}{R} \right) = \frac{1}{R^2} \frac{\partial}{\partial R} \left(R^2 \left(\frac{1}{R} \right) \right) + \frac{1}{R \sin \theta} \frac{\partial}{\partial \theta} (0 \sin \theta) + \frac{1}{R \sin \theta} \frac{\partial}{\partial \phi} 0 = \frac{1}{R^2}, \\
\nabla \times \mathbf{D} &= \nabla \times \left(\frac{\hat{\mathbf{R}}}{R} \right) \\
&= \hat{\mathbf{R}} \frac{1}{R \sin \theta} \left(\frac{\partial}{\partial \theta} (0 \sin \theta) - \frac{\partial}{\partial \phi} 0 \right) + \hat{\boldsymbol{\theta}} \frac{1}{R} \left(\frac{1}{\sin \theta} \frac{\partial}{\partial \phi} \left(\frac{1}{R} \right) - \frac{\partial}{\partial R} (R(0)) \right) \\
&\quad + \hat{\boldsymbol{\phi}} \frac{1}{R} \left(\frac{\partial}{\partial R} (R(0)) - \frac{\partial}{\partial \theta} \left(\frac{1}{R} \right) \right) = \hat{\mathbf{r}} 0 + \hat{\boldsymbol{\theta}} 0 + \hat{\boldsymbol{\phi}} 0.
\end{aligned}$$

The field \mathbf{D} is conservative but not solenoidal.

(e)

$$\begin{aligned}
\mathbf{E} &= \hat{\mathbf{r}} \left(3 - \frac{r}{1+r} \right) + \hat{\mathbf{z}} z, \\
\nabla \cdot \mathbf{E} &= \frac{1}{r} \frac{\partial}{\partial r} (r E_r) + \frac{1}{r} \frac{\partial E_\phi}{\partial \phi} + \frac{\partial E_z}{\partial z} \\
&= \frac{1}{r} \frac{\partial}{\partial r} \left(3r - \frac{r^2}{1+r} \right) + 1 \\
&= \frac{1}{r} \left[3 - \frac{2r}{1+r} + \frac{r^2}{(1+r)^2} \right] + 1 \\
&= \frac{1}{r} \left[\frac{3 + 3r^2 + 6r - 2r - 2r^2 + r^2}{(1+r)^2} \right] + 1 = \frac{2r^2 + 4r + 3}{r(1+r)^2} + 1 \neq 0, \\
\nabla \times \mathbf{E} &= \hat{\mathbf{r}} \left(\frac{1}{r} \frac{\partial E_z}{\partial \phi} - \frac{\partial E_\phi}{\partial z} \right) + \hat{\boldsymbol{\phi}} \left(\frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r} \right) + \hat{\mathbf{z}} \left(\frac{1}{r} \frac{\partial}{\partial r} (r E_\phi) - \frac{1}{r} \frac{\partial E_r}{\partial \phi} \right) = 0.
\end{aligned}$$

Hence, \mathbf{E} is conservative, but not solenoidal.

(f)

$$\begin{aligned}\mathbf{F} &= \frac{\hat{\mathbf{x}}y + \hat{\mathbf{y}}x}{x^2 + y^2} = \hat{\mathbf{x}} \frac{y}{x^2 + y^2} + \hat{\mathbf{y}} \frac{x}{x^2 + y^2}, \\ \nabla \cdot \mathbf{F} &= \frac{\partial}{\partial x} \left(\frac{y}{x^2 + y^2} \right) + \frac{\partial}{\partial y} \left(\frac{x}{x^2 + y^2} \right) \\ &= \frac{-2xy}{(x^2 + y^2)^2} + \frac{-2xy}{(x^2 + y^2)^2} \neq 0, \\ \nabla \times \mathbf{F} &= \hat{\mathbf{x}}(0 - 0) + \hat{\mathbf{y}}(0 - 0) + \hat{\mathbf{z}} \left[\frac{\partial}{\partial x} \left(\frac{x}{x^2 + y^2} \right) - \frac{\partial}{\partial y} \left(\frac{y}{x^2 + y^2} \right) \right] \\ &= \hat{\mathbf{z}} \left(\frac{1}{x^2 + y^2} - \frac{2x^2}{(x^2 + y^2)^2} - \frac{1}{x^2 + y^2} + \frac{2y^2}{(x^2 + y^2)^2} \right) \\ &= \hat{\mathbf{z}} \frac{2(y^2 - x^2)}{(x^2 + y^2)^2} \neq 0.\end{aligned}$$

Hence, \mathbf{F} is neither solenoidal nor conservative.

(g)

$$\begin{aligned}\mathbf{G} &= \hat{\mathbf{x}}(x^2 + z^2) - \hat{\mathbf{y}}(y^2 + x^2) - \hat{\mathbf{z}}(y^2 + z^2), \\ \nabla \cdot \mathbf{G} &= \frac{\partial}{\partial x}(x^2 + z^2) - \frac{\partial}{\partial y}(y^2 + x^2) - \frac{\partial}{\partial z}(y^2 + z^2) \\ &= 2x - 2y - 2z \neq 0, \\ \nabla \times \mathbf{G} &= \hat{\mathbf{x}} \left(-\frac{\partial}{\partial y}(y^2 + z^2) + \frac{\partial}{\partial z}(y^2 + x^2) \right) + \hat{\mathbf{y}} \left(\frac{\partial}{\partial z}(x^2 + z^2) + \frac{\partial}{\partial x}(y^2 + z^2) \right) \\ &\quad + \hat{\mathbf{z}} \left(-\frac{\partial}{\partial x}(y^2 + x^2) - \frac{\partial}{\partial y}(x^2 + z^2) \right) \\ &= -\hat{\mathbf{x}}2y + \hat{\mathbf{y}}2z - \hat{\mathbf{z}}2x \neq 0.\end{aligned}$$

Hence, \mathbf{G} is neither solenoidal nor conservative.

(h)

$$\begin{aligned}\mathbf{H} &= \hat{\mathbf{R}}(Re^{-R}), \\ \nabla \cdot \mathbf{H} &= \frac{1}{R^2} \frac{\partial}{\partial R} (R^3 e^{-R}) = \frac{1}{R^2} (3R^2 e^{-R} - R^3 e^{-R}) = e^{-R}(3 - R) \neq 0, \\ \nabla \times \mathbf{H} &= 0.\end{aligned}$$

Hence, \mathbf{H} is conservative, but not solenoidal.
