Problem 3.58 Find the Laplacian of the following scalar functions:

(a) $V_1 = 10r^3 \sin 2\phi$

(b) $V_2 = (2/R^2)\cos\theta \sin\phi$

Solution:

(a)

$$\begin{aligned}
\nabla^2 V_1 &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_1}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V_1}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} \\
&= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}(10r^3 \sin 2\phi)\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}(10r^3 \sin 2\phi) + 0 \\
&= \frac{1}{r}\frac{\partial}{\partial r}(30r^3 \sin 2\phi) - \frac{1}{r^2}(10r^3)4\sin 2\phi \\
&= 90r\sin 2\phi - 40r\sin 2\phi = 50r\sin 2\phi.
\end{aligned}$$

(b)

$$\begin{aligned}
\nabla^2 V_2 &= \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V_2}{\partial R}\right) + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V_2}{\partial \theta}\right) + \frac{1}{R^2\sin^2\theta}\frac{\partial^2 V_2}{\partial \phi^2} \\
&= \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial}{\partial R}\left(\frac{2}{R^2}\cos\theta\sin\phi\right)\right) \\
&\quad + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial \theta}\left(\frac{2}{R^2}\cos\theta\sin\phi\right)\right) \\
&\quad + \frac{1}{R^2\sin^2\theta}\frac{\partial^2}{\partial \phi^2}\left(\frac{2}{R^2}\cos\theta\sin\phi\right) \\
&= \frac{4}{R^4}\cos\theta\sin\phi - \frac{4}{R^4}\cos\theta\sin\phi - \frac{2}{R^4}\frac{\cos\theta}{\sin^2\theta}\sin\phi \\
&= -\frac{2}{R^4}\frac{\cos\theta\sin\phi}{\sin^2\theta}.
\end{aligned}$$