Problem 3.8 By expansion in Cartesian coordinates, prove:

(a) the relation for the scalar triple product given by (3.29), and

(b) the relation for the vector triple product given by (3.33).

Solution:

(a) Proof of the scalar triple product given by Eq. (3.29): From Eq. (3.27),

$$\begin{aligned}
\mathbf{A}\times\mathbf{B} &= \hat{\mathbf{x}}(A_yB_z - A_zB_y) + \hat{\mathbf{y}}(A_zB_x - A_xB_z) + \hat{\mathbf{z}}(A_xB_y - A_yB_x), \\
\mathbf{B}\times\mathbf{C} &= \hat{\mathbf{x}}(B_yC_z - B_zC_y) + \hat{\mathbf{y}}(B_zC_x - B_xC_z) + \hat{\mathbf{z}}(B_xC_y - B_yC_x), \\
\mathbf{C}\times\mathbf{A} &= \hat{\mathbf{x}}(C_yA_z - C_zA_y) + \hat{\mathbf{y}}(C_zA_x - C_xA_z) + \hat{\mathbf{z}}(C_xA_y - C_yA_x).
\end{aligned}$$

Employing Eq. (3.21), it is easily shown that

$$\begin{aligned}
\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) &= A_x(B_yC_z - B_zC_y) + A_y(B_zC_x - B_xC_z) + A_z(B_xC_y - B_yC_x), \\
\mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) &= B_x(C_yA_z - C_zA_y) + B_y(C_zA_x - C_xA_z) + B_z(C_xA_y - C_yA_x), \\
\mathbf{C}\cdot(\mathbf{A}\times\mathbf{B}) &= C_x(A_yB_z - A_zB_y) + C_y(A_zB_x - A_xB_z) + C_z(A_xB_y - A_yB_x),
\end{aligned}$$

which are all the same.

(b) Proof of the vector triple product given by Eq. (3.33): The evaluation of the left hand side employs the expression above for $\mathbf{B}\times\mathbf{C}$ with Eq. (3.27):

$$\begin{aligned}
\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) &= \mathbf{A}\times(\hat{\mathbf{x}}(B_yC_z - B_zC_y) + \hat{\mathbf{y}}(B_zC_x - B_xC_z) + \hat{\mathbf{z}}(B_xC_y - B_yC_x)) \\
&= \hat{\mathbf{x}}(A_y(B_xC_y - B_yC_x) - A_z(B_zC_x - B_xC_z)) \\
&\quad + \hat{\mathbf{y}}(A_z(B_yC_z - B_zC_y) - A_x(B_xC_y - B_yC_x)) \\
&\quad + \hat{\mathbf{z}}(A_x(B_zC_x - B_xC_z) - A_y(B_yC_z - B_zC_y)),
\end{aligned}$$

while the right hand side, evaluated with the aid of Eq. (3.21), is

$$\begin{aligned}
\mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B}) &= \mathbf{B}(A_xC_x + A_yC_y + A_zC_z) - \mathbf{C}(A_xB_x + A_yB_y + A_zB_z) \\
&= \hat{\mathbf{x}}(B_x(A_yC_y + A_zC_z) - C_x(A_yB_y + A_zB_z)) \\
&\quad + \hat{\mathbf{y}}(B_y(A_xC_x + A_zC_z) - C_y(A_xB_x + A_zB_z)) \\
&\quad + \hat{\mathbf{z}}(B_z(A_xC_x + A_yC_y) - C_z(A_xB_x + A_yB_y)).
\end{aligned}$$

By rearranging the expressions for the components, the left hand side is equal to the right hand side.