

Problem 4.23 Repeat Problem 4.22 for $\mathbf{D} = \hat{\mathbf{x}}xy^3z^3$ (C/m²).

Solution:

(a) From Eq. (4.26), $\rho_v = \nabla \cdot \mathbf{D} = \frac{\partial}{\partial x}(xy^3z^3) = y^3z^3$.

(b) Total charge Q is given by Eq. (4.27):

$$Q = \int_V \nabla \cdot \mathbf{D} \, dV = \int_{z=0}^2 \int_{y=0}^2 \int_{x=0}^2 y^3 z^3 \, dx \, dy \, dz = \frac{xy^4z^4}{16} \Big|_{x=0}^2 \Big|_{y=0}^2 \Big|_{z=0}^2 = 32 \text{ C.}$$

(c) Using Gauss' law we have

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = F_{\text{front}} + F_{\text{back}} + F_{\text{right}} + F_{\text{left}} + F_{\text{top}} + F_{\text{bottom}}.$$

Note that $\mathbf{D} = \hat{\mathbf{x}}D_x$, so only F_{front} and F_{back} (integration over $\hat{\mathbf{z}}$ surfaces) will contribute to the integral.

$$\begin{aligned} F_{\text{front}} &= \int_{z=0}^2 \int_{y=0}^2 (\hat{\mathbf{x}}xy^3z^3) \Big|_{x=2} \cdot (\hat{\mathbf{x}} \, dy \, dz) \\ &= \int_{z=0}^2 \int_{y=0}^2 xy^3z^3 \Big|_{x=2} \, dy \, dz = \left(2 \left(\frac{y^4z^4}{16} \right) \Big|_{y=0}^2 \right) \Big|_{z=0}^2 = 32, \\ F_{\text{back}} &= \int_{z=0}^2 \int_{y=0}^2 (\hat{\mathbf{x}}xy^3z^3) \Big|_{x=0} \cdot (-\hat{\mathbf{x}} \, dy \, dz) = - \int_{z=0}^2 \int_{y=0}^2 xy^3z^3 \Big|_{x=0} \, dy \, dz = 0. \end{aligned}$$

Thus $Q = \oint \mathbf{D} \cdot d\mathbf{s} = 32 + 0 + 0 + 0 + 0 + 0 = 32 \text{ C.}$
