

4.49 An infinitely long conducting cylinder of radius a has a surface charge density ρ_s . The cylinder is surrounded by a dielectric medium with $\epsilon_r = 4$ and contains no free charges. The tangential component of the electric field in the region $r \geq a$ is given by $\mathbf{E}_t = -\hat{\phi} \cos \phi / r^2$. Since a static conductor cannot have any tangential field, this must be cancelled by an externally applied electric field. Find the surface charge density on the conductor.

Solution: Let the conducting cylinder be medium 2 and the surrounding dielectric medium be medium 1. In medium 1,

$$\mathbf{E}_1 = \hat{\mathbf{r}} E_r - \hat{\phi} \frac{1}{r^2} \cos \phi,$$

with E_r , the normal component of \mathbf{E}_1 , unknown. The surface charge density is related to E_r . To find E_r , we invoke Gauss's law in medium 1:

$$\nabla \cdot \mathbf{D}_1 = 0,$$

or

$$\frac{1}{r} \frac{\partial}{\partial r} (r E_r) + \frac{1}{r} \frac{\partial}{\partial \phi} \left(-\frac{1}{r^2} \cos \phi \right) = 0,$$

which leads to

$$\frac{\partial}{\partial r} (r E_r) = \frac{\partial}{\partial \phi} \left(\frac{1}{r^2} \cos \phi \right) = -\frac{1}{r^2} \sin \phi.$$

Integrating both sides with respect to r ,

$$\begin{aligned} \int \frac{\partial}{\partial r} (r E_r) dr &= -\sin \phi \int \frac{1}{r^2} dr \\ r E_r &= \frac{1}{r} \sin \phi, \end{aligned}$$

or

$$E_r = \frac{1}{r^2} \sin \phi.$$

Hence,

$$\mathbf{E}_1 = \hat{\mathbf{r}} \frac{1}{r^2} \sin \phi - \hat{\phi} \frac{1}{r^2} \cos \phi.$$

According to Eq. (4.93),

$$\hat{\mathbf{n}}_2 \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s,$$

where $\hat{\mathbf{n}}_2$ is the normal to the boundary and points away from medium 1. Hence, $\hat{\mathbf{n}}_2 = \hat{\mathbf{r}}$. Also, $\mathbf{D}_2 = 0$ because the cylinder is a conductor. Consequently,

$$\rho_s = \hat{\mathbf{r}} \cdot \mathbf{D}_1|_{r=a}$$

$$\begin{aligned}
&= \hat{\mathbf{r}} \cdot \boldsymbol{\varepsilon}_1 \mathbf{E}_1|_{r=a} \\
&= \hat{\mathbf{r}} \cdot \boldsymbol{\varepsilon}_r \boldsymbol{\varepsilon}_0 \left[\hat{\mathbf{r}} \frac{1}{r^2} \sin \phi \right] \Big|_{r=a} \\
&= \frac{4\boldsymbol{\varepsilon}_0}{a^2} \sin \phi \quad (\text{C/m}^2).
\end{aligned}$$
