

Problem 4.63 Use the image method to find the capacitance per unit length of an infinitely long conducting cylinder of radius a situated at a distance d from a parallel conducting plane, as shown in Fig. P4.63.

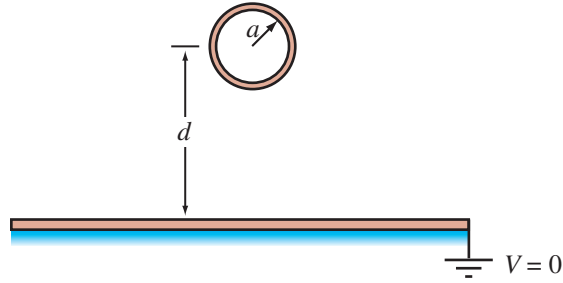


Figure P4.63: Conducting cylinder above a conducting plane (Problem 4.63).

Solution: Let us distribute charge ρ_l (C/m) on the conducting cylinder. Its image

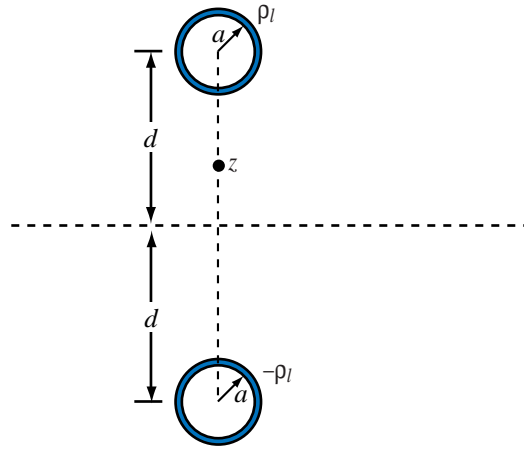


Figure P4.63: (a) Cylinder and its image.

cylinder at $z = -d$ will have charge density $-\rho_l$.

For the line at $z = d$, the electric field at any point z (at a distance of $d - z$ from the center of the cylinder) is, from Eq. (4.33),

$$\mathbf{E}_1 = -\hat{\mathbf{z}} \frac{\rho_l}{2\pi\epsilon_0(d - z)}$$

where $-\hat{\mathbf{z}}$ is the direction away from the cylinder. Similarly for the image cylinder at distance $(d+z)$ and carrying charge $-\rho_l$,

$$\mathbf{E}_2 = \hat{\mathbf{z}} \frac{(-\rho_l)}{2\pi\epsilon_0(d+z)} = -\hat{\mathbf{z}} \frac{\rho_l}{2\pi\epsilon_0(d+z)}.$$

The potential difference between the cylinders is obtained by integrating the total electric field from $z = -(d-a)$ to $z = (d-a)$:

$$\begin{aligned} V &= - \int_{-(d-a)}^{d-a} (\mathbf{E}_1 + \mathbf{E}_2) \cdot \hat{\mathbf{z}} dz \\ &= - \int_{-(d-a)}^{d-a} -\hat{\mathbf{z}} \frac{\rho_l}{2\pi\epsilon_0} \left(\frac{1}{d-z} + \frac{1}{d+z} \right) \cdot \hat{\mathbf{z}} dz \\ &= \frac{\rho_l}{2\pi\epsilon_0} \int_{-(d-a)}^{d-a} \left(\frac{1}{d-z} + \frac{1}{d+z} \right) dz \\ &= \frac{\rho_l}{2\pi\epsilon_0} [-\ln(d-z) + \ln(d+z)]_{-(d-a)}^{d-a} \\ &= \frac{\rho_l}{2\pi\epsilon_0} [-\ln(a) + \ln(2d-a) + \ln(2d-a) - \ln(a)] \\ &= \frac{\rho_l}{\pi\epsilon_0} \ln \left(\frac{2d-a}{a} \right). \end{aligned}$$

For a length L , $Q = \rho_l L$ and

$$C = \frac{Q}{V} = \frac{\rho_l L}{(\rho_l/\pi\epsilon_0) \ln[(2d-a)/a]},$$

and the capacitance per unit length is

$$C' = \frac{C}{L} = \frac{\pi\epsilon_0}{\ln[(2d/a) - 1]} \quad (\text{C/m}).$$