Problem 4.8 An electron beam shaped like a circular cylinder of radius r_0 carries a charge density given by

$$\rho_{\rm v} = \left(\frac{-\rho_0}{1+r^2}\right) \quad \text{(C/m}^3\text{)}$$

where ρ_0 is a positive constant and the beam's axis is coincident with the z-axis.

(a) Determine the total charge contained in length L of the beam.

(b) If the electrons are moving in the $+z$-direction with uniform speed u, determine the magnitude and direction of the current crossing the z-plane.

Solution:

(a)

$$\begin{aligned} Q &= \int_{r=0}^{r_0} \int_{z=0}^{L} \rho_{\rm v}\, d\mathscr{V} = \int_{r=0}^{r_0} \int_{z=0}^{L} \left(\frac{-\rho_0}{1+r^2}\right) 2\pi r\, dr\, dz \\ &= -2\pi\rho_0 L \int_0^{r_0} \frac{r}{1+r^2}\, dr = -\pi\rho_0 L \ln(1+r_0^2). \end{aligned}$$

(b)

$$\begin{aligned} \mathbf{J} &= \rho_{\rm v}\mathbf{u} = -\hat{\mathbf{z}} \frac{u\rho_0}{1+r^2} \quad \text{(A/m}^2\text{)}, \\ I &= \int \mathbf{J} \cdot d\mathbf{s} \\ &= \int_{r=0}^{r_0} \int_{\phi=0}^{2\pi} \left(-\hat{\mathbf{z}} \frac{u\rho_0}{1+r^2}\right) \cdot \hat{\mathbf{z}} r\, dr\, d\phi \\ &= -2\pi u\rho_0 \int_0^{r_0} \frac{r}{1+r^2}\, dr = -\pi u\rho_0 \ln(1+r_0^2) \quad \text{(A)}. \end{aligned}$$

Current direction is along $-\hat{\mathbf{z}}$.