Problem 5.10 An infinitely long, thin conducting sheet defined over the space $0 \le x \le w$ and $-\infty \le y \le \infty$ is carrying a current with a uniform surface current density $\mathbf{J}_s = \hat{\mathbf{y}}5$ (A/m). Obtain an expression for the magnetic field at point $P = (0,0,z)$ in Cartesian coordinates.

Solution:

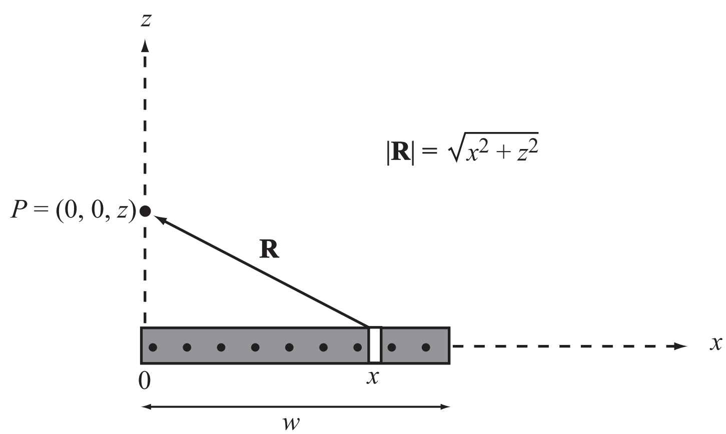

Figure P5.10: Conducting sheet of width w in x–y plane.

The sheet can be considered to be a large number of infinitely long but narrow wires each dx wide lying next to each other, with each carrying a current $I_x = J_s\,dx$. The wire at a distance x from the origin is at a distance vector $\mathbf{R}$ from point P, with

$$\mathbf{R} = -\hat{\mathbf{x}}x + \hat{\mathbf{z}}z.$$

Equation (5.30) provides an expression for the magnetic field due to an infinitely long wire carrying a current I as

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} = \frac{\hat{\boldsymbol{\phi}} I}{2\pi r}.$$

We now need to adapt this expression to the present situation by replacing I with $I_x = J_s\,dx$, replacing r with $R = (x^2+z^2)^{1/2}$, as shown in Fig. P5.10, and by assigning the proper direction for the magnetic field. From the Biot–Savart law, the direction of $\mathbf{H}$ is governed by $\mathbf{l} \times \mathbf{R}$, where $\mathbf{l}$ is the direction of current flow. In the present case, $\mathbf{l}$ is in the $\hat{\mathbf{y}}$ direction. Hence, the direction of the field is

$$\frac{\mathbf{l} \times \mathbf{R}}{|\mathbf{l} \times \mathbf{R}|} = \frac{\hat{\mathbf{y}} \times (-\hat{\mathbf{x}}x + \hat{\mathbf{z}}z)}{|\hat{\mathbf{y}} \times (-\hat{\mathbf{x}}x + \hat{\mathbf{z}}z)|} = \frac{\hat{\mathbf{x}}z + \hat{\mathbf{z}}x}{(x^2+z^2)^{1/2}}.$$

Therefore, the field $d\mathbf{H}$ due to the current I_x is

$$d\mathbf{H} = \frac{\hat{\mathbf{x}}z + \hat{\mathbf{z}}x}{(x^2+z^2)^{1/2}} \, \frac{I_x}{2\pi R} = \frac{(\hat{\mathbf{x}}z + \hat{\mathbf{z}}x)J_s\,dx}{2\pi(x^2+z^2)}\,,$$

and the total field is

$$\begin{aligned}
\mathbf{H}(0,0,z) &= \int_{x=0}^{w} (\hat{\mathbf{x}}z + \hat{\mathbf{z}}x)\frac{J_s\,dx}{2\pi(x^2+z^2)} \\
&= \frac{J_s}{2\pi}\int_{x=0}^{w} (\hat{\mathbf{x}}z + \hat{\mathbf{z}}x)\frac{dx}{x^2+z^2} \\
&= \frac{J_s}{2\pi}\left(\hat{\mathbf{x}}z\int_{x=0}^{w}\frac{dx}{x^2+z^2} + \hat{\mathbf{z}}\int_{x=0}^{w}\frac{x\,dx}{x^2+z^2}\right) \\
&= \frac{J_s}{2\pi}\left(\hat{\mathbf{x}}z\left(\frac{1}{z}\tan^{-1}\left(\frac{x}{z}\right)\right)\Bigg|_{x=0}^{w} + \hat{\mathbf{z}}\left(\tfrac{1}{2}\ln(x^2+z^2)\right)\Big|_{x=0}^{w}\right) \\
&= \frac{5}{2\pi}\left[\hat{\mathbf{x}}2\pi\tan^{-1}\left(\frac{w}{z}\right) + \hat{\mathbf{z}}\tfrac{1}{2}(\ln(w^2+z^2) - \ln(0+z^2))\right] \quad \text{for } z \neq 0, \\
&= \frac{5}{2\pi}\left[\hat{\mathbf{x}}2\pi\tan^{-1}\left(\frac{w}{z}\right) + \hat{\mathbf{z}}\tfrac{1}{2}\ln\left(\frac{w^2+z^2}{z^2}\right)\right] \quad \text{(A/m)} \quad \text{for } z \neq 0.
\end{aligned}$$

An alternative approach is to employ Eq. (5.24a) directly.