

Problem 5.18 An infinitely long, thin conducting sheet of width w along the x -direction lies in the x - y plane and carries a current I in the $-y$ -direction. Determine the following:

- (a) The magnetic field at a point P midway between the edges of the sheet and at a height h above it (Fig. P5.18).
- (b) The force per unit length exerted on an infinitely long wire passing through point P and parallel to the sheet if the current through the wire is equal in magnitude but opposite in direction to that carried by the sheet.

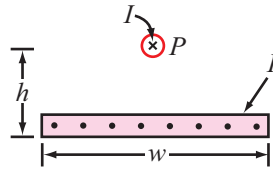


Figure P5.18: A linear current source above a current sheet (Problem 5.18).

Solution:

(a) The sheet can be considered to consist of a large number of infinitely long but narrow wires each dx wide lying next to each other, with each carrying a current $I_x = I dx/w$. If we choose the coordinate system shown in Fig. P5.18, the wire at a distance x from the origin is at a distance vector \mathbf{R} from point P , with

$$\mathbf{R} = -\hat{\mathbf{x}}x + \hat{\mathbf{z}}h.$$

Equation (5.30) provides an expression for the magnetic field due to an infinitely long wire carrying a current I as

$$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} = \hat{\phi} \frac{I}{2\pi r}.$$

We now need to adapt this expression to the present situation by replacing I with $I_x = I dx/w$, replacing r with $R = (x^2 + h^2)^{1/2}$, and by assigning the proper direction for the magnetic field. From the Biot-Savart law, the direction of \mathbf{H} is governed by $\mathbf{l} \times \mathbf{R}$, where \mathbf{l} is the direction of current flow. In the present case, \mathbf{l} is out of the page, which is the $-\hat{\mathbf{y}}$ direction. Hence, the direction of the field is

$$\frac{\mathbf{l} \times \mathbf{R}}{|\mathbf{l} \times \mathbf{R}|} = \frac{-\hat{\mathbf{y}} \times (-\hat{\mathbf{x}}x + \hat{\mathbf{z}}h)}{|-\hat{\mathbf{y}} \times (-\hat{\mathbf{x}}x + \hat{\mathbf{z}}h)|} = \frac{-(\hat{\mathbf{x}}h + \hat{\mathbf{z}}x)}{(x^2 + h^2)^{1/2}}.$$

Therefore, the field $d\mathbf{H}$ due to current I_x is

$$d\mathbf{H} = \frac{-(\hat{\mathbf{x}}h + \hat{\mathbf{z}}x)}{(x^2 + h^2)^{1/2}} \frac{I_x}{2\pi R} = \frac{-(\hat{\mathbf{x}}h + \hat{\mathbf{z}}x)I dx}{2\pi w(x^2 + h^2)^{3/2}},$$

and the total field is

$$\begin{aligned}
\mathbf{H}(0,0,h) &= \int_{x=-w/2}^{w/2} -(\hat{\mathbf{x}}h + \hat{\mathbf{z}}x) \frac{I dx}{2\pi w(x^2 + h^2)} \\
&= \frac{-I}{2\pi w} \int_{x=-w/2}^{w/2} (\hat{\mathbf{x}}h + \hat{\mathbf{z}}x) \frac{dx}{x^2 + h^2} \\
&= \frac{-I}{2\pi w} \left(\hat{\mathbf{x}}h \int_{x=-w/2}^{w/2} \frac{dx}{x^2 + h^2} + \hat{\mathbf{z}} \int_{x=-w/2}^{w/2} \frac{x dx}{x^2 + h^2} \right) \\
&= \frac{-I}{2\pi w} \left(\hat{\mathbf{x}}h \left(\frac{1}{h} \tan^{-1} \left(\frac{x}{h} \right) \right) \Big|_{x=-w/2}^{w/2} + \hat{\mathbf{z}} \left(\frac{1}{2} \ln(x^2 + h^2) \right) \Big|_{x=-w/2}^{w/2} \right) \\
&= -\hat{\mathbf{x}} \frac{I}{\pi w} \tan^{-1} \left(\frac{w}{2h} \right) \quad (\text{A/m}).
\end{aligned}$$

At P in Fig. P5.18, the field is pointing to the left. The z -component could have been assumed zero with a symmetry argument. An alternative solution is to employ Eq. (5.24a) directly.

(b) From Eq. (5.9), a differential force is of the form $d\mathbf{F}_m = I d\mathbf{l} \times \mathbf{B}$ or, assuming $d\mathbf{l} = \hat{\mathbf{a}}_\ell d\ell$, the force per unit length is given by

$$\mathbf{F}'_m = \frac{\partial \mathbf{F}_m}{\partial \ell} = I \hat{\mathbf{a}}_\ell \times \mathbf{B} = I \hat{\mathbf{y}} \times \left(-\hat{\mathbf{x}} \frac{\mu_0 I}{\pi w} \tan^{-1} \left(\frac{w}{2h} \right) \right) = \hat{\mathbf{z}} \frac{\mu_0 I^2}{\pi w} \tan^{-1} \left(\frac{w}{2h} \right) \quad (\text{N}).$$

The force is repulsive; the wire is experiencing a force pushing it up.
