Problem 5.27 In a given region of space, the vector magnetic potential is given by $\mathbf{A} = \hat{\mathbf{x}}5\cos\pi y + \hat{\mathbf{z}}(2+\sin\pi x)$ (Wb/m).

(a) Determine **B**.

(b) Use Eq. (5.66) to calculate the magnetic flux passing through a square loop with 0.25-m-long edges if the loop is in the x–y plane, its center is at the origin, and its edges are parallel to the x- and y-axes.

(c) Calculate Φ again using Eq. (5.67).

Solution:

(a) From Eq. (5.53), $\mathbf{B} = \nabla\times\mathbf{A} = \hat{\mathbf{z}}5\pi\sin\pi y - \hat{\mathbf{y}}\pi\cos\pi x$.

(b) From Eq. (5.66),

$$\begin{aligned}\Phi = \iint \mathbf{B}\cdot d\mathbf{s} &= \int_{y=-0.125\text{ m}}^{0.125\text{ m}} \int_{x=-0.125\text{ m}}^{0.125\text{ m}} (\hat{\mathbf{z}}5\pi\sin\pi y - \hat{\mathbf{y}}\pi\cos\pi x)\cdot(\hat{\mathbf{z}}\,dx\,dy) \\ &= \left(\left(-5\pi x\frac{\cos\pi y}{\pi}\right)\Big|_{x=-0.125}^{0.125}\right)\Big|_{y=-0.125}^{0.125} \\ &= \frac{-5}{4}\left(\cos\left(\frac{\pi}{8}\right) - \cos\left(\frac{-\pi}{8}\right)\right) = 0.\end{aligned}$$

(c) From Eq. (5.67), $\Phi = \oint_C \mathbf{A}\cdot d\boldsymbol{\ell}$, where C is the square loop in the x-y plane with sides of length 0.25 m centered at the origin. Thus, the integral can be written as

$$\Phi = \oint_C \mathbf{A}\cdot d\boldsymbol{\ell} = S_{\text{front}} + S_{\text{back}} + S_{\text{left}} + S_{\text{right}},$$

where S_{front}, S_{back}, S_{left}, and S_{right} are the sides of the loop.

$$\begin{aligned}S_{\text{front}} &= \int_{x=-0.125}^{0.125} (\hat{\mathbf{x}}5\cos\pi y + \hat{\mathbf{z}}(2+\sin\pi x))|_{y=-0.125}\cdot(\hat{\mathbf{x}}\,dx) \\ &= \int_{x=-0.125}^{0.125} 5\cos\pi y|_{y=-0.125}\,dx \\ &= \left((5x\cos\pi y)|_{y=-0.125}\right)\Big|_{x=-0.125}^{0.125} = \frac{5}{4}\cos\left(\frac{-\pi}{8}\right) = \frac{5}{4}\cos\left(\frac{\pi}{8}\right), \\ S_{\text{back}} &= \int_{x=-0.125}^{0.125} (\hat{\mathbf{x}}5\cos\pi y + \hat{\mathbf{z}}(2+\sin\pi x))|_{y=0.125}\cdot(-\hat{\mathbf{x}}\,dx) \\ &= -\int_{x=-0.125}^{0.125} 5\cos\pi y|_{y=0.125}\,dx \\ &= \left((-5x\cos\pi y)|_{y=0.125}\right)\Big|_{x=-0.125}^{0.125} = -\frac{5}{4}\cos\left(\frac{\pi}{8}\right),\end{aligned}$$

$$
\begin{aligned}
S_{\text{left}} &= \int_{y=-0.125}^{0.125} \left(\hat{\mathbf{x}}5\cos\pi y + \hat{\mathbf{z}}(2+\sin\pi x)\right)\big|_{x=-0.125} \cdot (-\hat{\mathbf{y}}\,dy) \\
&= -\int_{y=-0.125}^{0.125} 0\big|_{x=-0.125}\,dy = 0, \\
S_{\text{right}} &= \int_{y=-0.125}^{0.125} \left(\hat{\mathbf{x}}5\cos\pi y + \hat{\mathbf{z}}(2+\sin\pi x)\right)\big|_{x=0.125} \cdot (\hat{\mathbf{y}}\,dy) \\
&= \int_{y=-0.125}^{0.125} 0\big|_{x=0.125}\,dy = 0.
\end{aligned}
$$

Thus,

$$
\Phi = \oint_c \mathbf{A}\cdot d\boldsymbol{\ell} = S_{\text{front}} + S_{\text{back}} + S_{\text{left}} + S_{\text{right}} = \frac{5}{4}\cos\left(\frac{\pi}{8}\right) - \frac{5}{4}\cos\left(\frac{\pi}{8}\right) + 0 + 0 = 0.
$$