Problem 5.28 A uniform current density given by

$$\mathbf{J} = \hat{\mathbf{z}} J_0 \quad (\text{A/m}^2)$$

gives rise to a vector magnetic potential

$$\mathbf{A} = -\hat{\mathbf{z}} \frac{\mu_0 J_0}{4}(x^2 + y^2) \quad (\text{Wb/m})$$

(a) Apply the vector Poisson's equation to confirm the above statement.

(b) Use the expression for **A** to find **H**.

(c) Use the expression for **J** in conjunction with Ampère's law to find **H**. Compare your result with that obtained in part (b).

Solution:

(a)

$$\begin{aligned}
\nabla^2 \mathbf{A} = \hat{\mathbf{x}} \nabla^2 A_x + \hat{\mathbf{y}} \nabla^2 A_y + \hat{\mathbf{z}} \nabla^2 A_z &= \hat{\mathbf{z}} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \left[-\mu_0 \frac{J_0}{4} (x^2 + y^2) \right] \\
&= -\hat{\mathbf{z}} \mu_0 \frac{J_0}{4} (2+2) = -\hat{\mathbf{z}} \mu_0 J_0 .
\end{aligned}$$

Hence, $\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}$ is verified.

(b)

$$\begin{aligned}
\mathbf{H} = \frac{1}{\mu_0} \nabla \times \mathbf{A} &= \frac{1}{\mu_0} \left[\hat{\mathbf{x}} \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \hat{\mathbf{y}} \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \hat{\mathbf{z}} \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \right] \\
&= \frac{1}{\mu_0} \left(\hat{\mathbf{x}} \frac{\partial A_z}{\partial y} - \hat{\mathbf{y}} \frac{\partial A_z}{\partial x} \right) \\
&= \frac{1}{\mu_0} \left[\hat{\mathbf{x}} \frac{\partial}{\partial y} \left(-\mu_0 \frac{J_0}{4} (x^2 + y^2) \right) - \hat{\mathbf{y}} \frac{\partial}{\partial x} \left(-\mu_0 \frac{J_0}{4} (x^2 + y^2) \right) \right] \\
&= -\hat{\mathbf{x}} \frac{J_0 y}{2} + \hat{\mathbf{y}} \frac{J_0 x}{2} \quad (\text{A/m}).
\end{aligned}$$

(c)

$$\begin{aligned}
\oint_C \mathbf{H} \cdot d\mathbf{l} = I &= \int_S \mathbf{J} \cdot d\mathbf{s}, \\
\hat{\boldsymbol{\phi}} H_\phi \cdot \hat{\boldsymbol{\phi}} 2\pi r &= J_0 \cdot \pi r^2, \\
\mathbf{H} = \hat{\boldsymbol{\phi}} H_\phi &= \hat{\boldsymbol{\phi}} J_0 \frac{r}{2} .
\end{aligned}$$

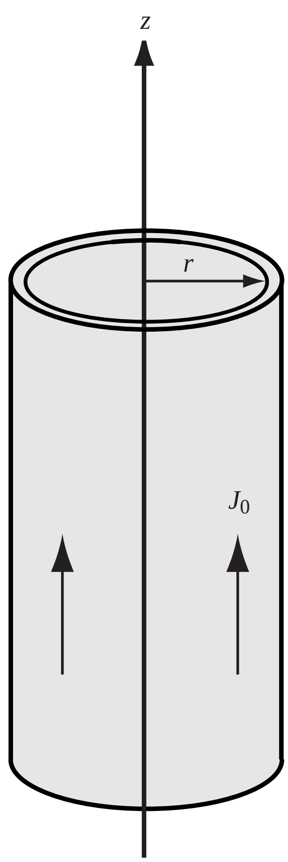

Figure P5.28: Current cylinder of Problem 5.28.

We need to convert the expression from cylindrical to Cartesian coordinates. From Table 3-2,

$$\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi = -\hat{\mathbf{x}}\frac{y}{\sqrt{x^2+y^2}} + \hat{\mathbf{y}}\frac{x}{\sqrt{x^2+y^2}}\,,$$

$$r = \sqrt{x^2+y^2}\,.$$

Hence

$$\mathbf{H} = \left(-\hat{\mathbf{x}}\frac{y}{\sqrt{x^2+y^2}} + \hat{\mathbf{y}}\frac{x}{\sqrt{x^2+y^2}}\right)\cdot\frac{J_0}{2}\sqrt{x^2+y^2} = -\hat{\mathbf{x}}\frac{yJ_0}{2} + \hat{\mathbf{y}}\frac{xJ_0}{2}\,,$$

which is identical with the result of part (b).