Problem 5.41 Determine the mutual inductance between the circular loop and the linear current shown in Fig. P5.41.

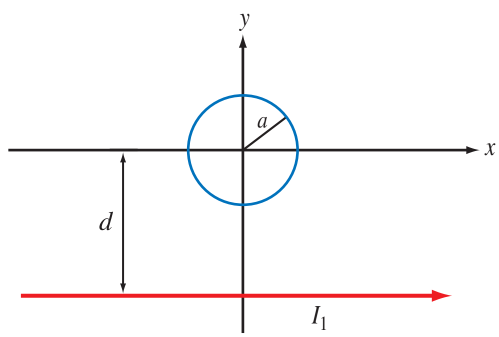

Figure P5.41: Linear conductor with current I_1 next to a circular loop of radius a at distance d (Problem 5.41).

Solution: To calculate the magnetic flux through the loop due to the current in the conductor, we consider a thin strip of thickness dy at location y, as shown. The magnetic field is the same at all points across the strip because they are all equidistant (at $r = d + y$) from the linear conductor. The magnetic flux through the strip is

$$\begin{aligned} d\Phi_{12} &= \mathbf{B}(y) \cdot d\mathbf{s} = \hat{\mathbf{z}} \frac{\mu_0 I}{2\pi(d+y)} \cdot \hat{\mathbf{z}}\, 2(a^2 - y^2)^{1/2}\, dy \\ &= \frac{\mu_0 I (a^2 - y^2)^{1/2}}{\pi(d+y)}\, dy \end{aligned}$$

$$\begin{aligned} L_{12} &= \frac{1}{I} \int_S d\Phi_{12} \\ &= \frac{\mu_0}{\pi} \int_{y=-a}^{a} \frac{(a^2 - y^2)^{1/2}\, dy}{(d+y)} \end{aligned}$$

Let $z = d + y \;\rightarrow\; dz = dy$. Hence,

$$\begin{aligned} L_{12} &= \frac{\mu_0}{\pi} \int_{z=d-a}^{d+a} \frac{\sqrt{a^2 - (z-d)^2}}{z}\, dz \\ &= \frac{\mu_0}{\pi} \int_{d-a}^{d+a} \frac{\sqrt{(a^2 - d^2) + 2dz - z^2}}{z}\, dz \\ &= \frac{\mu_0}{\pi} \int \frac{\sqrt{R}}{z}\, dz \end{aligned}$$

where $R = a_0 + b_0 z + c_0 z^2$ and

$$\begin{aligned} a_0 &= a^2 - d^2 \\ b_0 &= 2d \\ c_0 &= -1 \\ \Delta &= 4a_0c_0 - b_0^2 = -4a^2 < 0 \end{aligned}$$

From Gradshteyn and Ryzhik, *Table of Integrals, Series, and Products* (Academic Press, 1980, p. 84), we have

$$\int \frac{\sqrt{R}}{z}\, dz = \sqrt{R} + a_0 \int \frac{dz}{z\sqrt{R}} + \frac{b_0}{z} \int \frac{dz}{\sqrt{R}}\,.$$

For

$$\sqrt{R}\Big|_{z=d-a}^{d+a} = \sqrt{a^2 - d^2 + 2dz - z^2}\Big|_{z=d-a}^{d+a} = 0 - 0 = 0.$$

For $\int \frac{dz}{z\sqrt{R}}$, several solutions exist depending on the sign of a_0 and Δ.

For this problem, $\Delta < 0$, also let $a_0 < 0$ (i.e., $d > a$). Using the table of integrals,

$$\begin{aligned} a_0 \int \frac{dz}{z\sqrt{R}} &= a_0 \left[\frac{1}{\sqrt{-a_0}} \sin^{-1}\left(\frac{2a_0 + b_0 z}{z\sqrt{b_0^2 - 4a_0c_0}} \right) \right]_{z=d-a}^{d+a} \\ &= -\sqrt{d^2 - a^2} \left[\sin^{-1}\left(\frac{a^2 - d^2 + dz}{az} \right) \right]_{z=d-a}^{d+a} \\ &= -\pi\sqrt{d^2 - a^2}\,. \end{aligned}$$

For $\int \frac{dz}{\sqrt{R}}$, different solutions exist depending on the sign of c_0 and Δ.

In this problem, $\Delta < 0$ and $c_0 < 0$. From the table of integrals,

$$\begin{aligned} \frac{b_0}{z} \int \frac{dz}{\sqrt{R}} &= \frac{b_0}{2} \left[\frac{-1}{\sqrt{-c_0}} \sin^{-1} \frac{2c_0 z + b_0}{\sqrt{-\Delta}} \right]_{z=d-a}^{d+a} \\ &= -d \left[\sin^{-1}\left(\frac{d-z}{a} \right) \right]_{z=d-a}^{d+a} = \pi d. \end{aligned}$$

Thus

$$\begin{aligned} L_{12} &= \frac{\mu_0}{\pi} \cdot \left[\pi d - \pi\sqrt{d^2 - a^2} \right] \\ &= \mu_0 \left[d - \sqrt{d^2 - a^2} \right]. \end{aligned}$$