

Problem 6.27 A Hertzian dipole is a short conducting wire carrying an approximately constant current over its length l . If such a dipole is placed along the z -axis with its midpoint at the origin, and if the current flowing through it is $i(t) = I_0 \cos \omega t$, find the following:

- (a) The retarded vector potential $\tilde{\mathbf{A}}(R, \theta, \phi)$ at an observation point $Q(R, \theta, \phi)$ in a spherical coordinate system.
- (b) The magnetic field phasor $\tilde{\mathbf{H}}(R, \theta, \phi)$.

Assume l to be sufficiently small so that the observation point is approximately equidistant to all points on the dipole; that is, assume $R' \simeq R$.

Solution:

(a) In phasor form, the current is given by $\tilde{I} = I_0$. Explicitly writing the volume integral in Eq. (6.84) as a double integral over the wire cross section and a single integral over its length,

$$\tilde{\mathbf{A}} = \frac{\mu}{4\pi} \int_{-l/2}^{l/2} \iint_s \frac{\tilde{\mathbf{J}}(\mathbf{R}_i) e^{-jkR'}}{R'} ds dz,$$

where s is the wire cross section. The wire is infinitesimally thin, so that R' is not a function of x or y and the integration over the cross section of the wire applies only to the current density. Recognizing that $\tilde{\mathbf{J}} = \hat{\mathbf{z}} I_0 / s$, and employing the relation $R' \approx R$,

$$\tilde{\mathbf{A}} = \hat{\mathbf{z}} \frac{\mu I_0}{4\pi} \int_{-l/2}^{l/2} \frac{e^{-jkR'}}{R'} dz \approx \hat{\mathbf{z}} \frac{\mu I_0}{4\pi} \int_{-l/2}^{l/2} \frac{e^{-jkR}}{R} dz = \hat{\mathbf{z}} \frac{\mu I_0 l}{4\pi R} e^{-jkR}.$$

In spherical coordinates, $\hat{\mathbf{z}} = \hat{\mathbf{R}} \cos \theta - \hat{\boldsymbol{\theta}} \sin \theta$, and therefore

$$\tilde{\mathbf{A}} = (\hat{\mathbf{R}} \cos \theta - \hat{\boldsymbol{\theta}} \sin \theta) \frac{\mu I_0 l}{4\pi R} e^{-jkR}.$$

(b) From Eq. (6.85),

$$\begin{aligned} \tilde{\mathbf{H}} &= \frac{1}{\mu} \nabla \times \tilde{\mathbf{A}} = \frac{I_0 l}{4\pi} \nabla \times \left[(\hat{\mathbf{R}} \cos \theta - \hat{\boldsymbol{\theta}} \sin \theta) \frac{e^{-jkR}}{R} \right] \\ &= \frac{I_0 l}{4\pi} \hat{\boldsymbol{\phi}} \frac{1}{R} \left(\frac{\partial}{\partial R} (-\sin \theta e^{-jkR}) - \frac{\partial}{\partial \theta} \left(\cos \theta \frac{e^{-jkR}}{R} \right) \right) \\ &= \hat{\boldsymbol{\phi}} \frac{I_0 l \sin \theta e^{-jkR}}{4\pi R} \left(jk + \frac{1}{R} \right). \end{aligned}$$
