**Problem 6.29** The magnetic field in a given dielectric medium is given by

$$\mathbf{H} = \hat{\mathbf{y}}\, 6 \cos 2z \sin(2 \times 10^7 t - 0.1x) \quad \text{(A/m)},$$

where $x$ and $z$ are in meters. Determine:

**(a)** $\mathbf{E}$,

**(b)** the displacement current density $\mathbf{J}_\text{d}$, and

**(c)** the charge density $\rho_\text{v}$.

**Solution:**

**(a)**

$$\begin{aligned}
\mathbf{H} &= \hat{\mathbf{y}}\, 6 \cos 2z \sin(2 \times 10^7 t - 0.1x) = \hat{\mathbf{y}}\, 6 \cos 2z \cos(2 \times 10^7 t - 0.1x - \pi/2), \\
\widetilde{\mathbf{H}} &= \hat{\mathbf{y}}\, 6 \cos 2z\, e^{-j0.1x} e^{-j\pi/2} = -\hat{\mathbf{y}}\, j6 \cos 2z\, e^{-j0.1x}, \\
\widetilde{\mathbf{E}} &= \frac{1}{j\omega\varepsilon} \nabla \times \widetilde{\mathbf{H}} \\
&= \frac{1}{j\omega\varepsilon} \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & -j6 \cos 2z\, e^{-j0.1x} & 0 \end{vmatrix} \\
&= \frac{1}{j\omega\varepsilon} \left\{ \hat{\mathbf{x}} \left[ -\frac{\partial}{\partial z} (-j6 \cos 2z\, e^{-j0.1x}) \right] + \hat{\mathbf{z}} \left[ \frac{\partial}{\partial x} (-j6 \cos 2z\, e^{-j0.1x}) \right] \right\} \\
&= \hat{\mathbf{x}} \left( -\frac{12}{\omega\varepsilon} \sin 2z\, e^{-j0.1x} \right) + \hat{\mathbf{z}} \left( \frac{j0.6}{\omega\varepsilon} \cos 2z\, e^{-j0.1x} \right).
\end{aligned}$$

From the given expression for $\mathbf{H}$,

$$\begin{aligned}
\omega &= 2 \times 10^7 \quad \text{(rad/s)}, \\
\beta &= 0.1 \quad \text{(rad/m)}.
\end{aligned}$$

Hence,

$$u_\text{p} = \frac{\omega}{\beta} = 2 \times 10^8 \quad \text{(m/s)},$$

and

$$\varepsilon_\text{r} = \left( \frac{c}{u_\text{p}} \right)^2 = \left( \frac{3 \times 10^8}{2 \times 10^8} \right)^2 = 2.25.$$

Using the values for $\omega$ and $\varepsilon$, we have

$$\begin{aligned}
\widetilde{\mathbf{E}} &= (-\hat{\mathbf{x}}\, 30 \sin 2z + \hat{\mathbf{z}}\, j1.5 \cos 2z) \times 10^3 e^{-j0.1x} \quad \text{(V/m)}, \\
\mathbf{E} &= \left[ -\hat{\mathbf{x}}\, 30 \sin 2z \cos(2 \times 10^7 t - 0.1x) - \hat{\mathbf{z}}\, 1.5 \cos 2z \sin(2 \times 10^7 t - 0.1x) \right] \quad \text{(kV/m)}.
\end{aligned}$$

**(b)**

$$\widetilde{\mathbf{D}} = \varepsilon\widetilde{\mathbf{E}} = \varepsilon_{\mathrm{r}}\varepsilon_0\widetilde{\mathbf{E}} = (-\hat{\mathbf{x}}\,0.6\sin 2z + \hat{\mathbf{z}}\,j0.03\cos 2z) \times 10^{-6}e^{-j0.1x} \quad (\mathrm{C/m^2}),$$

$$\mathbf{J}_{\mathrm{d}} = \frac{\partial \mathbf{D}}{\partial t},$$

or

$$\widetilde{\mathbf{J}}_{\mathrm{d}} = j\omega\widetilde{\mathbf{D}} = (-\hat{\mathbf{x}}\,j12\sin 2z - \hat{\mathbf{z}}\,0.6\cos 2z)e^{-j0.1x},$$

$$\begin{aligned}\mathbf{J}_{\mathrm{d}} &= \mathfrak{Re}[\widetilde{\mathbf{J}}_{\mathrm{d}}e^{j\omega t}] \\ &= \left[\hat{\mathbf{x}}\,12\sin 2z\sin(2\times 10^7 t - 0.1x) - \hat{\mathbf{z}}\,0.6\cos 2z\cos(2\times 10^7 t - 0.1x)\right] \quad (\mathrm{A/m^2}).\end{aligned}$$

**(c)** We can find $\rho_{\mathrm{v}}$ from

$$\nabla\cdot\mathbf{D} = \rho_{\mathrm{v}}$$

or from

$$\nabla\cdot\mathbf{J} = -\frac{\partial\rho_{\mathrm{v}}}{\partial t}.$$

Applying Maxwell's equation,

$$\rho_{\mathrm{v}} = \nabla\cdot\mathbf{D} = \varepsilon\nabla\cdot\mathbf{E} = \varepsilon_{\mathrm{r}}\varepsilon_0\left(\frac{\partial E_x}{\partial x} + \frac{\partial E_z}{\partial z}\right)$$

yields

$$\begin{aligned}\rho_{\mathrm{v}} &= \varepsilon_{\mathrm{r}}\varepsilon_0\left\{\frac{\partial}{\partial x}\left[-30\sin 2z\cos(2\times 10^7 t - 0.1x)\right]\right. \\ &\qquad \left. + \frac{\partial}{\partial z}\left[-1.5\cos 2z\sin(2\times 10^7 t - 0.1x)\right]\right\} \\ &= \varepsilon_{\mathrm{r}}\varepsilon_0\left[-3\sin 2z\sin(2\times 10^7 t - 0.1x) + 3\sin 2z\sin(2\times 10^7 t - 0.1x)\right] = 0.\end{aligned}$$