Problem 8.29 A plane wave in air with

$$\widetilde{\mathbf{E}}^{\mathrm{i}} = (\hat{\mathbf{x}}9 - \hat{\mathbf{y}}4 - \hat{\mathbf{z}}6)e^{-j(2x+3z)} \quad \text{(V/m)}$$

is incident upon the planar surface of a dielectric material, with $\varepsilon_{\mathrm{r}} = 2.25$, occupying the half-space $z \geq 0$. Determine

(a) The incidence angle θ_{i}.

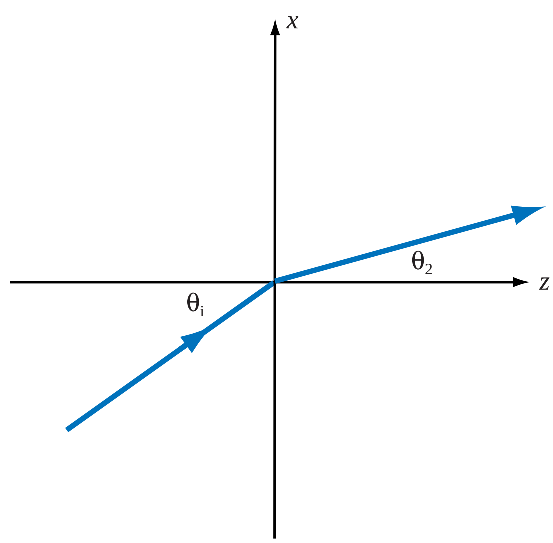
(b) The frequency of the wave.

(c) The field $\widetilde{\mathbf{E}}^{\mathrm{r}}$ of the reflected wave.

(d) The field $\widetilde{\mathbf{E}}^{\mathrm{t}}$ of the wave transmitted into the dielectric medium.

(e) The average power density carried by the wave into the dielectric medium.

Solution:



(a) From the exponential of the given expression, it is clear that the wave direction of travel is in the x–z plane. By comparison with the expressions in (8.48a) for perpendicular polarization or (8.65a) for parallel polarization, both of which have the same phase factor, we conclude that:

$$k_1 \sin\theta_{\mathrm{i}} = 2,$$
$$k_1 \cos\theta_{\mathrm{i}} = 3.$$

Hence,

$$k_1 = \sqrt{2^2 + 3^2} = 3.6 \quad \text{(rad/m)}$$
$$\theta_\text{i} = \tan^{-1}(2/3) = 33.7^\circ.$$

Also,

$$k_2 = k_1\sqrt{\varepsilon_{\text{r}_2}} = 3.6\sqrt{2.25} = 5.4 \quad \text{(rad/m)}$$
$$\theta_2 = \sin^{-1}\left[\sin\theta_\text{i}\sqrt{\frac{1}{2.25}}\right] = 21.7^\circ.$$

(b)

$$k_1 = \frac{2\pi f}{c}$$
$$f = \frac{k_1 c}{2\pi} = \frac{3.6 \times 3 \times 10^8}{2\pi} = 172 \text{ MHz}.$$

(c) In order to determine the electric field of the reflected wave, we first have to determine the polarization of the wave. The vector argument in the given expression for $\widetilde{\mathbf{E}}^\text{i}$ indicates that the incident wave is a mixture of parallel and perpendicular polarization components. Perpendicular polarization has a $\hat{\mathbf{y}}$-component only (see 8.46a), whereas parallel polarization has only $\hat{\mathbf{x}}$ and $\hat{\mathbf{z}}$ components (see 8.65a). Hence, we shall decompose the incident wave accordingly:

$$\widetilde{\mathbf{E}}^\text{i} = \widetilde{\mathbf{E}}^\text{i}_\perp + \widetilde{\mathbf{E}}^\text{i}_\parallel$$

with

$$\widetilde{\mathbf{E}}^\text{i}_\perp = -\hat{\mathbf{y}}\,4e^{-j(2x+3z)} \quad \text{(V/m)}$$
$$\widetilde{\mathbf{E}}^\text{i}_\parallel = (\hat{\mathbf{x}}9 - \hat{\mathbf{z}}6)e^{-j(2x+3z)} \quad \text{(V/m)}$$

From the above expressions, we deduce:

$$E^\text{i}_{\perp 0} = -4 \text{ V/m}$$
$$E^\text{i}_{\parallel 0} = \sqrt{9^2 + 6^2} = 10.82 \text{ V/m}.$$

Next, we calculate Γ and τ for each of the two polarizations:

$$\Gamma_\perp = \frac{\cos\theta_\text{i} - \sqrt{(\varepsilon_2/\varepsilon_1) - \sin^2\theta_\text{i}}}{\cos\theta_\text{i} + \sqrt{(\varepsilon_2/\varepsilon_1) - \sin^2\theta_\text{i}}}$$

Using $\theta_\mathrm{i} = 33.7^\circ$ and $\varepsilon_2/\varepsilon_1 = 2.25/1 = 2.25$ leads to:

$$\begin{aligned}
\Gamma_\perp &= -0.25 \\
\tau_\perp &= 1 + \Gamma_\perp = 0.75.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\Gamma_\perp &= \frac{-(\varepsilon_2/\varepsilon_1)\cos\theta_\mathrm{i} + \sqrt{(\varepsilon_2/\varepsilon_1) - \sin^2\theta_\mathrm{i}}}{(\varepsilon_2/\varepsilon_1)\cos\theta_\mathrm{i} + \sqrt{(\varepsilon_2/\varepsilon_1) - \sin^2\theta_\mathrm{i}}} = -0.15, \\
\tau_\parallel &= (1 + \Gamma_\parallel)\frac{\cos\theta_\mathrm{i}}{\cos\theta_\mathrm{t}} = (1 - 0.15)\frac{\cos 33.7^\circ}{\cos 21.7^\circ} = 0.76.
\end{aligned}$$

The electric fields of the reflected and transmitted waves for the two polarizations are given by (8.49a), (8.49c), (8.65c), and (8.65e):

$$\begin{aligned}
\widetilde{\mathbf{E}}^\mathrm{r}_\perp &= \hat{\mathbf{y}}E^\mathrm{r}_{\perp 0}e^{-jk_1(x\sin\theta_\mathrm{r} - z\cos\theta_\mathrm{r})} \\
\widetilde{\mathbf{E}}^\mathrm{t}_\perp &= \hat{\mathbf{y}}E^\mathrm{t}_{\perp 0}e^{-jk_2(x\sin\theta_\mathrm{t} + z\cos\theta_\mathrm{t})} \\
\widetilde{\mathbf{E}}^\mathrm{r}_\parallel &= (\hat{\mathbf{x}}\cos\theta_\mathrm{r} + \hat{\mathbf{z}}\sin\theta_\mathrm{r})E^\mathrm{r}_{\parallel 0}e^{-jk_1(x\sin\theta_\mathrm{r} - z\cos\theta_\mathrm{r})} \\
\widetilde{\mathbf{E}}^\mathrm{t}_\parallel &= (\hat{\mathbf{x}}\cos\theta_\mathrm{t} - \hat{\mathbf{z}}\sin\theta_\mathrm{t})E^\mathrm{t}_{\parallel 0}e^{-jk_2(x\sin\theta_\mathrm{t} + z\cos\theta_\mathrm{t})}
\end{aligned}$$

Based on our earlier calculations:

$$\begin{aligned}
\theta_\mathrm{r} &= \theta_\mathrm{i} = 33.7^\circ \\
\theta_\mathrm{t} &= 21.7^\circ \\
k_1 &= 3.6 \text{ rad/m}, \qquad k_2 = 5.4 \text{ rad/m}, \\
E^\mathrm{r}_{\perp 0} &= \Gamma_\perp E^\mathrm{i}_{\perp 0} = (-0.25) \times (-4) = 1 \text{ V/m}. \\
E^\mathrm{t}_{\perp 0} &= \tau_\perp E^\mathrm{i}_{\perp 0} = 0.75 \times (-4) = -3 \text{ V/m}. \\
E^\mathrm{r}_{\parallel 0} &= \Gamma_\parallel E^\mathrm{i}_{\parallel 0} = (-0.15) \times 10.82 = -1.62 \text{ V/m}. \\
E^\mathrm{t}_{\parallel 0} &= \tau_\parallel E^\mathrm{i}_{\parallel 0} = 0.76 \times 10.82 = 8.22 \text{ V/m}.
\end{aligned}$$

Using the above values, we have:

$$\begin{aligned}
\widetilde{\mathbf{E}}^\mathrm{r} &= \widetilde{\mathbf{E}}^\mathrm{r}_\perp + \widetilde{\mathbf{E}}^\mathrm{r}_\parallel \\
&= (\hat{\mathbf{x}}E^\mathrm{r}_{\parallel 0}\cos\theta_\mathrm{r} + \hat{\mathbf{y}}E^\mathrm{r}_{\perp 0} + \hat{\mathbf{z}}E^\mathrm{r}_{\parallel 0}\sin\theta_\mathrm{r})e^{-j(2x-3z)} \\
&= (-\hat{\mathbf{x}}1.35 + \hat{\mathbf{y}} - \hat{\mathbf{z}}0.90)e^{-j(2x-3z)} \quad \text{(V/m)}.
\end{aligned}$$

(d)

$$\begin{aligned}\widetilde{\mathbf{E}}^{\mathrm{t}} &= \widetilde{\mathbf{E}}^{\mathrm{t}}_{\perp} + \widetilde{\mathbf{E}}^{\mathrm{t}}_{\parallel} \\ &= (\hat{\mathbf{x}}7.65 - \hat{\mathbf{y}}3 - \hat{\mathbf{z}}3.05)e^{-j(2x+5z)} \quad \text{(V/m)}.\end{aligned}$$

(e)

$$\begin{aligned}S^{\mathrm{t}} &= \frac{|E_0^{\mathrm{t}}|^2}{2\eta_2} \\ |E_0^{\mathrm{t}}|^2 &= (7.65)^2 + 3^2 + (3.05)^2 = 76.83 \\ \eta_2 &= \frac{\eta_0}{\sqrt{\varepsilon_{\mathrm{r}_2}}} = \frac{377}{1.5} = 251.3\ \Omega \\ S^{\mathrm{t}} &= \frac{76.83}{2 \times 251.3} = 152.86 \quad (\mathrm{mW/m^2}).\end{aligned}$$