

Problem 8.42 For a rectangular waveguide operating in the TE₁₀ mode, obtain expressions for the surface charge density $\tilde{\rho}_s$ and surface current density $\tilde{\mathbf{J}}_s$ on each of the four walls of the guide.

Solution:

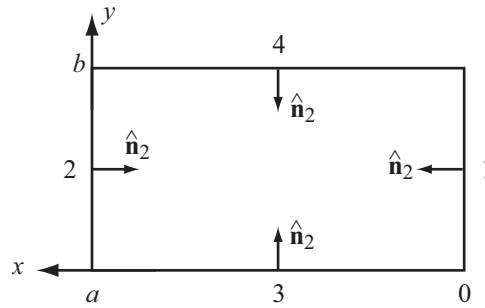
For TE₁₀, the expressions for $\tilde{\mathbf{E}}$ and $\tilde{\mathbf{H}}$ are given by Eq. (8.110) with $m = 1$ and $n = 0$,

$$\begin{aligned}\tilde{E}_x &= 0, \\ \tilde{E}_y &= -j \frac{\omega \mu \pi H_0}{k_c^2 a} \sin\left(\frac{\pi x}{a}\right) e^{-j\beta z}, \\ \tilde{E}_z &= 0, \\ \tilde{H}_x &= j \frac{\beta \pi H_0}{k_c^2 a} \sin\left(\frac{\pi x}{a}\right) e^{-j\beta z}, \\ \tilde{H}_y &= 0, \\ \tilde{H}_z &= H_0 \cos\left(\frac{\pi x}{a}\right) e^{-j\beta z}.\end{aligned}$$

The applicable boundary conditions are given in Table 6-2. At the boundary between a dielectric (medium 1) and a conductor (medium 2),

$$\begin{aligned}\tilde{\rho}_s &= \hat{\mathbf{n}}_2 \cdot \tilde{\mathbf{D}}_1 = \epsilon_1 \hat{\mathbf{n}}_2 \cdot \tilde{\mathbf{E}}_1, \\ \tilde{\mathbf{J}}_s &= \hat{\mathbf{n}}_2 \times \tilde{\mathbf{H}}_1,\end{aligned}$$

where $\tilde{\mathbf{E}}_1$ and $\tilde{\mathbf{H}}_1$ are the fields inside the guide, ϵ_1 is the permittivity of the material filling the guide, and $\hat{\mathbf{n}}_2$ is the normal to the guide wall, pointing away from the wall (inwardly). In view of the coordinate system defined for the guide, $\hat{\mathbf{n}}_2 = \hat{\mathbf{x}}$ for side wall at $x = 0$, $\hat{\mathbf{n}}_2 = -\hat{\mathbf{x}}$ for wall at $x = a$, etc.



(a) At side wall 1 at $x = 0$, $\hat{\mathbf{n}}_2 = \hat{\mathbf{x}}$. Hence,

$$\begin{aligned}\rho_s &= \varepsilon_1 \hat{\mathbf{x}} \cdot \hat{\mathbf{y}} E_y|_{x=0} = 0 \\ \mathbf{J}_s &= \hat{\mathbf{x}} \times (\hat{\mathbf{x}} \tilde{H}_x + \hat{\mathbf{z}} \tilde{H}_z)|_{x=0} \\ &= -\hat{\mathbf{y}} \tilde{H}_z|_{x=0} \\ &= -\hat{\mathbf{y}} H_0 e^{-j\beta z}.\end{aligned}$$

(b) At side wall 2 at $x = a$, $\hat{\mathbf{n}}_2 = -\hat{\mathbf{x}}$. Hence,

$$\begin{aligned}\rho_s &= 0 \\ \mathbf{J}_s &= \hat{\mathbf{y}} H_0 e^{-j\beta z}.\end{aligned}$$

(c) At bottom surface at $y = 0$, $\hat{\mathbf{n}}_2 = \hat{\mathbf{y}}$. Hence,

$$\begin{aligned}\rho_s &= \varepsilon_1 \hat{\mathbf{y}} \cdot \hat{\mathbf{y}} E_y|_{y=0} \\ &= -j \frac{\omega \varepsilon \mu \pi H_0}{k_c^2 a} \sin\left(\frac{\pi x}{a}\right) e^{-j\beta z} \\ \tilde{\mathbf{J}}_s &= \hat{\mathbf{y}} \times (\hat{\mathbf{x}} \tilde{H}_x + \hat{\mathbf{z}} \tilde{H}_z) \\ &= H_0 \left[\hat{\mathbf{x}} \cos\left(\frac{\pi x}{a}\right) - \hat{\mathbf{z}} j \frac{\beta \pi}{k_c^2 a} \sin\left(\frac{\pi x}{a}\right) \right] e^{-j\beta z}.\end{aligned}$$

(d) At top surface at $y = b$, $\hat{\mathbf{n}}_2 = -\hat{\mathbf{y}}$. Hence,

$$\begin{aligned}\tilde{\rho}_s &= j \frac{\omega \varepsilon \mu \pi H_0}{k_c^2 a} \sin\left(\frac{\pi x}{a}\right) e^{-j\beta z} \\ \tilde{\mathbf{J}}_s &= H_0 \left[-\hat{\mathbf{x}} \cos\left(\frac{\pi x}{a}\right) + \hat{\mathbf{z}} j \frac{\beta \pi}{k_c^2 a} \sin\left(\frac{\pi x}{a}\right) \right] e^{-j\beta z}.\end{aligned}$$
