3.47 For the vector field $\mathbf{E} = \hat{\mathbf{r}}10e^{-r} - \hat{\mathbf{z}}3z$, verify the divergence theorem for the cylindrical region enclosed by $r = 2$, $z = 0$, and $z = 4$.

Solution:

$$\begin{aligned}
\oint \vec{E} \cdot d\vec{s} &= \int_{r=0}^{2} \int_{\phi=0}^{2\pi} ((\hat{\mathbf{r}}10\exp -r - \hat{\mathbf{z}}3z) \cdot (-\hat{\mathbf{z}}r\,dr\,d\phi))|_{z=0} \\
&\quad + \int_{\phi=0}^{2\pi} \int_{z=0}^{4} ((\hat{\mathbf{r}}10\exp -r - \hat{\mathbf{z}}3z) \cdot (\hat{\mathbf{r}}r\,d\phi\,dz))|_{r=2} \\
&\quad + \int_{r=0}^{2} \int_{\phi=0}^{2\pi} ((\hat{\mathbf{r}}10\exp -r - \hat{\mathbf{z}}3z) \cdot (\hat{\mathbf{z}}r\,dr\,d\phi))|_{z=4} \\
&= 0 + \int_{\phi=0}^{2\pi} \int_{z=0}^{4} 10\exp -22\,d\phi\,dz + \int_{r=0}^{2} \int_{\phi=0}^{2\pi} -12r\,dr\,d\phi \\
&= 160\pi\exp -2 - 48\pi \approx -82.77,
\end{aligned}$$

$$\begin{aligned}
\iiint \nabla\cdot\vec{E}\,d\mathcal{V} &= \int_{z=0}^{4} \int_{r=0}^{2} \int_{\phi=0}^{2\pi} \left(\frac{10\exp -r(1-r)}{r} - 3 \right) r\,d\phi\,dr\,dz \\
&= 8\pi \int_{r=0}^{2} (10\exp -r(1-r) - 3r)\,dr \\
&= 8\pi \left(-10\exp -r + 10\exp -r(1+r) - \frac{3r^2}{2} \right) \Bigg|_{r=0}^{2} \\
&= 160\pi\exp -2 - 48\pi \approx -82.77.
\end{aligned}$$