4.49 An infinitely long conducting cylinder of radius a is surrounded by a dielectric medium that contains no free charges. If the tangential component of the electric field in the region $r \geq a$ is given by $\mathbf{E}_t = -\hat{\boldsymbol{\phi}} \cos\phi / r^2$, find $\mathbf{E}$ in that region.

Solution: Let the conducting cylinder be medium 1 and the surrounding dielectric medium be medium 2. In medium 2,

$$\mathbf{E}_2 = \hat{\mathbf{r}} E_r - \hat{\boldsymbol{\phi}} \frac{1}{r^2} \cos\phi,$$

with E_r, the normal component of $\mathbf{E}_2$, unknown. We invoke Gauss's law in medium 2:

$$\nabla \cdot \mathbf{D}_2 = 0,$$

or

$$\frac{1}{r} \frac{\partial}{\partial r} (r E_r) + \frac{1}{r} \frac{\partial}{\partial \phi} \left(-\frac{1}{r^2} \cos\phi \right) = 0,$$

which leads to

$$\frac{\partial}{\partial r} (r E_r) = \frac{\partial}{\partial \phi} \left(\frac{1}{r^2} \cos\phi \right) = -\frac{1}{r^2} \sin\phi.$$

Integrating both sides with respect to r,

$$\int \frac{\partial}{\partial r} (r E_r) \, dr = -\sin\phi \int \frac{1}{r^2} \, dr$$

$$r E_r = \frac{1}{r} \sin\phi,$$

or

$$E_r = \frac{1}{r^2} \sin\phi.$$

Hence,

$$\mathbf{E}_2 = \hat{\mathbf{r}} \frac{1}{r^2} \sin\phi - \hat{\boldsymbol{\phi}} \frac{1}{r^2} \cos\phi.$$