4.58 The capacitor shown in Fig. P4.58 consists of two parallel dielectric layers. Use energy considerations to show that the equivalent capacitance of the overall capacitor, C, is equal to the series combination of the capacitances of the individual layers, C_1 and C_2, namely

$$C = \frac{C_1 C_2}{C_1 + C_2} \tag{22}$$

where

$$C_1 = \varepsilon_1 \frac{A}{d_1}, \qquad C_2 = \varepsilon_2 \frac{A}{d_2}$$

(a) Let V_1 and V_2 be the electric potentials across the upper and lower dielectrics, respectively. What are the corresponding electric fields E_1 and E_2? By applying the appropriate boundary condition at the interface between the two dielectrics, obtain explicit expressions for E_1 and E_2 in terms of ε_1, ε_2, V, and the indicated dimensions of the capacitor.

(b) Calculate the energy stored in each of the dielectric layers and then use the sum to obtain an expression for C.

(c) Show that C is given by Eq. (26).

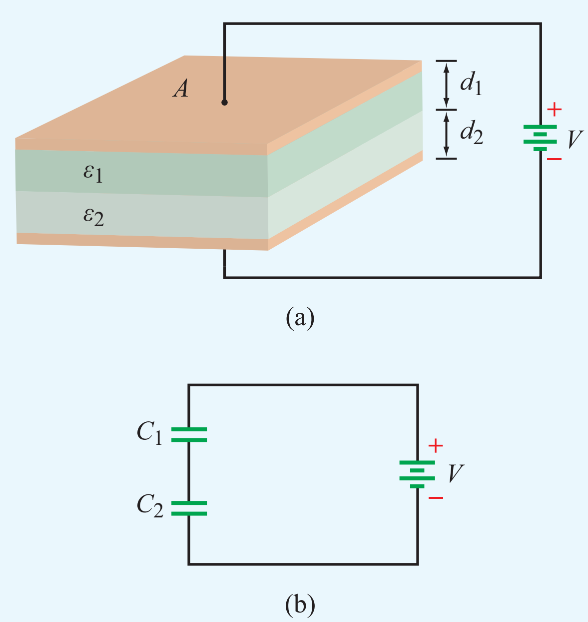

Figure P4.58 (a) Capacitor with parallel dielectric layers, and (b) equivalent circuit (Problem 4.58).

Solution:

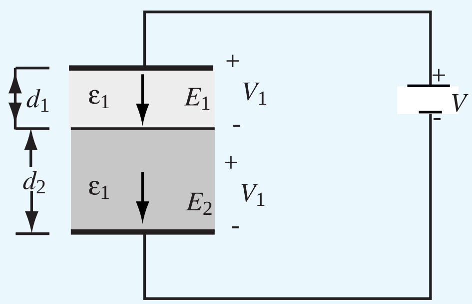

Figure P4.58 (c) Electric fields inside of capacitor.

(a) If V_1 is the voltage across the top layer and V_2 across the bottom layer, then

$$V = V_1 + V_2,$$

and

$$E_1 = \frac{V_1}{d_1}, \qquad E_2 = \frac{V_2}{d_2}.$$

According to boundary conditions, the normal component of **D** is continuous across the boundary (in the absence of surface charge). This means that at the interface between the two dielectric layers,

$$D_{1n} = D_{2n}$$

or

$$\varepsilon_1 E_1 = \varepsilon_2 E_2.$$

Hence,

$$V = E_1 d_1 + E_2 d_2 = E_1 d_1 + \frac{\varepsilon_1 E_1}{\varepsilon_2} d_2,$$

which can be solved for E_1:

$$E_1 = \frac{V}{d_1 + \dfrac{\varepsilon_1}{\varepsilon_2} d_2}.$$

Similarly,

$$E_2 = \frac{V}{d_2 + \dfrac{\varepsilon_2}{\varepsilon_1} d_1}.$$

(b)

$$W_{e_1} = \frac{1}{2}\varepsilon_1 E_1^2 \cdot \mathscr{V}_1 = \frac{1}{2}\varepsilon_1 \left(\frac{V}{d_1 + \dfrac{\varepsilon_1}{\varepsilon_2} d_2}\right)^2 \cdot A d_1 = \frac{1}{2} V^2 \left[\frac{\varepsilon_1 \varepsilon_2^2 A d_1}{(\varepsilon_2 d_1 + \varepsilon_1 d_2)^2}\right],$$

$$W_{e_2} = \frac{1}{2}\varepsilon_2 E_2^2 \cdot \mathscr{V}_2 = \frac{1}{2}\varepsilon_2 \left(\frac{V}{d_2 + \dfrac{\varepsilon_2}{\varepsilon_1} d_1}\right)^2 \cdot A d_2 = \frac{1}{2} V^2 \left[\frac{\varepsilon_1^2 \varepsilon_2 A d_2}{(\varepsilon_1 d_2 + \varepsilon_2 d_1)^2}\right],$$

$$W_e = W_{e_1} + W_{e_2} = \frac{1}{2} V^2 \left[\frac{\varepsilon_1 \varepsilon_2^2 A d_1 + \varepsilon_1^2 \varepsilon_2 A d_2}{(\varepsilon_1 d_2 + \varepsilon_2 d_1)^2}\right].$$

But $W_e = \frac{1}{2} C V^2$, hence,

$$C = \frac{\varepsilon_1 \varepsilon_2^2 A d_1 + \varepsilon_1^2 \varepsilon_2 A d_2}{(\varepsilon_2 d_1 + \varepsilon_1 d_2)^2} = \varepsilon_1 \varepsilon_2 A \frac{(\varepsilon_2 d_1 + \varepsilon_1 d_2)}{(\varepsilon_2 d_1 + \varepsilon_1 d_2)^2} = \frac{\varepsilon_1 \varepsilon_2 A}{\varepsilon_2 d_1 + \varepsilon_1 d_2}.$$

(c) Multiplying numerator and denominator of the expression for C by $A/d_1 d_2$, we have

$$C = \frac{\dfrac{\varepsilon_1 A}{d_1} \cdot \dfrac{\varepsilon_2 A}{d_2}}{\dfrac{\varepsilon_1 A}{d_1} + \dfrac{\varepsilon_2 A}{d_2}} = \frac{C_1 C_2}{C_1 + C_2},$$

where

$$C_1 = \frac{\varepsilon_1 A}{d_1}, \qquad C_2 = \frac{\varepsilon_2 A}{d_2}.$$