4.61 With reference to Fig. P4.61, charge Q is located at a distance d above a grounded half-plane located in the x–y plane and at a distance d from another grounded half-plane in the x–z plane. Use the image method to

(a) Establish the magnitudes, polarities, and locations of the images of charge Q with respect to each of the two ground planes (as if each is infinite in extent).

(b) Find the electric potential and electric field at an arbitrary point $P = (0, y, z)$.

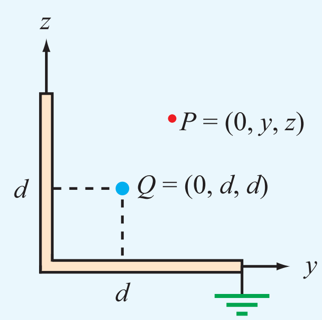

Figure P4.61 Charge Q next to two perpendicular, grounded, conducting half-planes.

Solution:

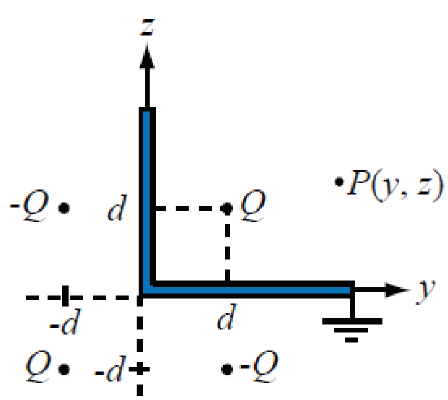

Figure P4.61 (a) Image charges.

(a) The original charge has magnitude and polarity $+Q$ at location $(0, d, d)$. Since the negative y-axis is shielded from the region of interest, there might as well be a conducting half-plane extending in the $-y$ direction as well as the $+y$ direction. This ground plane gives rise to an image charge of magnitude and polarity $-Q$ at location $(0,d,-d)$. In addition, since charges exist on the conducting half plane in the $+z$ direction, an image of this conducting half plane also appears in the $-z$ direction. This ground plane in the x-z plane gives rise to the image charges of $-Q$ at $(0,-d,d)$ and $+Q$ at $(0,-d,-d)$.

(b) Using Eq. (4.47) with $N=4$,

$$
\begin{aligned}
V(x,y,z) &= \frac{Q}{4\pi\varepsilon}\left(\frac{1}{|\hat{\mathbf{x}}x+\hat{\mathbf{y}}(y-d)+\hat{\mathbf{z}}(z-d)|}-\frac{1}{|\hat{\mathbf{x}}x+\hat{\mathbf{y}}(y+d)+\hat{\mathbf{z}}(z-d)|}\right.\\
&\qquad\left.+\frac{1}{|\hat{\mathbf{x}}x+\hat{\mathbf{y}}(y+d)+\hat{\mathbf{z}}(z+d)|}-\frac{1}{|\hat{\mathbf{x}}x+\hat{\mathbf{y}}(y-d)+\hat{\mathbf{z}}(z+d)|}\right)\\
&= \frac{Q}{4\pi\varepsilon}\left(\frac{1}{\sqrt{x^2+(y-d)^2+(z-d)^2}}-\frac{1}{\sqrt{x^2+(y+d)^2+(z-d)^2}}\right.\\
&\qquad\left.+\frac{1}{\sqrt{x^2+(y+d)^2+(z+d)^2}}-\frac{1}{\sqrt{x^2+(y-d)^2+(z+d)^2}}\right)\\
&= \frac{Q}{4\pi\varepsilon}\left(\frac{1}{\sqrt{x^2+y^2-2yd+z^2-2zd+2d^2}}\right.\\
&\qquad-\frac{1}{\sqrt{x^2+y^2+2yd+z^2-2zd+2d^2}}\\
&\qquad+\frac{1}{\sqrt{x^2+y^2+2yd+z^2+2zd+2d^2}}\\
&\qquad\left.-\frac{1}{\sqrt{x^2+y^2-2yd+z^2+2zd+2d^2}}\right) \qquad \text{(V)}.
\end{aligned}
$$

From Eq. (4.51),

$$
\begin{aligned}
\vec{E} &= -\nabla V\\
&= \frac{Q}{4\pi\varepsilon}\left(\nabla\frac{1}{\sqrt{x^2+(y-d)^2+(z-d)^2}}-\nabla\frac{1}{\sqrt{x^2+(y+d)^2+(z-d)^2}}\right.\\
&\qquad\left.+\nabla\frac{1}{\sqrt{x^2+(y+d)^2+(z+d)^2}}-\nabla\frac{1}{\sqrt{x^2+(y-d)^2+(z+d)^2}}\right)
\end{aligned}
$$

$$= \frac{Q}{4\pi\varepsilon}\left(\frac{\hat{\mathbf{x}}x+\hat{\mathbf{y}}(y-d)+\hat{\mathbf{z}}(z-d)}{\left(x^2+(y-d)^2+(z-d)^2\right)^{3/2}}-\frac{\hat{\mathbf{x}}x+\hat{\mathbf{y}}(y+d)+\hat{\mathbf{z}}(z-d)}{\left(x^2+(y+d)^2+(z-d)^2\right)^{3/2}}\right.$$

$$\left.+\frac{\hat{\mathbf{x}}x+\hat{\mathbf{y}}(y+d)+\hat{\mathbf{z}}(z+d)}{\left(x^2+(y+d)^2+(z+d)^2\right)^{3/2}}-\frac{\hat{\mathbf{x}}x+\hat{\mathbf{y}}(y-d)+\hat{\mathbf{z}}(z+d)}{\left(x^2+(y-d)^2+(z+d)^2\right)^{3/2}}\right) \qquad \text{(V/m)}.$$